

Pricing variable annuities with multi-layer expense strategy

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Abstract

We study the problem of pricing variable annuities with a multi-layer expense strategy, under which the insurer charges fees from the policyholder's account only when the account value lies in some pre-specified disjoint intervals, where on each pre-specified interval, the fee rate is fixed and can be different from that on other interval. We model the asset that is the underlying fund of the variable annuity by a hyper-exponential jump diffusion process. Theoretically, for a jump diffusion process with hyper-exponential jumps and three-valued drift, we obtain expressions for the Laplace transforms of its distribution and its occupation times, i.e., the time that it spends below or above a pre-specified level. With these results, we derive closed-form formulas to determine the fair fee rate. Moreover, the total fees that will be collected by the insurer and the total time of deducting fees are also computed. In addition, some numerical examples are presented to illustrate our results.

Keywords: Variable annuities; Multi-layer expense strategy; Hyper-exponential jump diffusion process; Laplace transform; Occupation times.

1. Introduction

A variable annuity (VA) is an equity-linked life insurance product with minimum guarantee on the death or maturity benefits. Generally, the insured chooses a mutual fund according to his/her risk preference, and contributes an initial premium to invest in the chosen fund. A fascinating feature of VAs, which makes them different from mutual funds, is that they usually provide some minimum guaranteed payoffs. There are many kinds of guarantees embedded in VAs (see, e.g., Bauer et al. (2008)) and many papers investigating the problem of pricing and hedging VAs, see for example, Gerber and Shiu (2003) and Bacinello et al. (2011). In this paper, we focus on the pricing of a variable annuity with level Guaranteed Minimum Maturity Benefits (GMMBs). But we

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remark that our results can be extended to price a variable annuity with level Guaranteed Minimum Death Benefits (GMDBs).

In general, for a VA, fees for the provided guarantees are deducted continuously from the policyholder's account during its lifetime by a fixed rate. If the VA expires earlier than expected (one main cause is that the policyholder lapses the policy), it is possible that fees collected by the insurer may be not enough to cover the guarantees. Therefore, how to reduce the surrender rate is an important question to the insurer. In Bauer et al. (2008), the authors have noted that a fixed fee rate will produce incentives for the policyholder to lapse the policy. The reason is that when the account value is higher, the guarantees (like put options) are out-of-the-money whereas the insured pays more fees. Thus, designing some proper fee charging method may reduce the possibility of surrendering the contract.

In Bernard et al. (2014a), under the simple Black-Scholes model, the authors proposed a state-dependent fee structure, under which fees are charged only if the account value is smaller than a pre-specified level. Mathematically, if we denote by F_t the value of the account at time t , then the total expenses charged by the insurer from time 0 to t are given by

$$\int_0^t \alpha_1 F_s \mathbf{1}_{\{F_s < B_1\}} ds, \quad (1.1)$$

where B_1 is the pre-specified level. Similar fee charging strategy has also been considered in Zhou and Wu (2015) under the double exponential jump diffusion process. This kind of fee deducting approach has some advantages. For example, it can avoid effectively the problem that the policyholder lapses the policy due to the high fees when the guarantees are deep out-of-the-money. However, as the insurer cannot charge fees when the policyholder's account value becomes large, the fair fee rate α_1^* obtained is too high to be used in practice, see numerical results in section 4 of Bernard et al. (2014a), section 5 of Delong (2014) and section 4 of Zhou and Wu (2015).

From the investigation on optimal surrender for VAs (see Bernard et al. (2014b) for example), we know that the optimal surrender strategy under a fixed fee rate structure (i.e., $B_1 = \infty$ in (1.1)) is a threshold strategy, i.e., the policyholder will lapse the policy when the value of the account exceeds a time-dependent barrier. Although the optimal surrender strategy for $B_1 < \infty$ has not been investigated sufficiently, it is very likely that the policyholder will surrender the policy when the value of the account becomes large enough. This leads to the following problem: if the market rises continuously to the surrender barrier (just after the inception of the policy) and the insured lapses the policy, then the insurer will have a few (or even no) fees income under the strategy (1.1).

In this paper, we extend the research in Bernard et al. (2014a) and Zhou and Wu (2015), and consider a multi-layer fee collecting method, under which the insurer charges expenses not only when the account value is lower than some pre-specified level but also when the account value exceeds some pre-specified

level. That is, the total expenses deducted until time t are given by

$$\int_0^t (\alpha_1 F_s \mathbf{1}_{\{F_s < B_1\}} + \alpha_2 F_s \mathbf{1}_{\{F_s \geq B_2\}}) ds, \quad (1.2)$$

where $B_1 \leq B_2$ and usually $\alpha_2 < \alpha_1$. As formula (1.2) contains the term $\alpha_1 F_s \mathbf{1}_{\{F_s < B_1\}}$, the fee strategy (1.2) inherits some advantage of (1.1). Moreover, the second term $\alpha_2 F_s \mathbf{1}_{\{F_s \geq B_2\}}$ can alleviate the problem discussed in the last paragraph. This is because when F_t exceeds B_2 , this term can reduce the return of the policyholder's account by α_2 , which makes the account value exceed the surrender barrier with less probability. In addition, numerical examples in section 5 illustrate that the fair fee rates α_1^* and α_2^* are reasonable.

We should mention that Delong (2014) has considered a general state-dependent fee structure (which includes (1.2)) under a general Lévy process. However, the fee rate obtained from the pricing principle (4.4) in his paper may yield arbitrage opportunities under some Lévy process (e.g., a hyper-exponential jump diffusion process considered in this paper), see the discussion presented after formula (4.5) in Delong (2014). More importantly, we have derived formulas for the Laplace transforms of the distribution and the occupation times of a jump diffusion process with three-valued drift and hyper-exponential jumps. Our approach is remarkable and can be applied to more complicated fee structure (see Remark 2.1).

The reminder of this paper is organized as follows. Our model and an important preliminary result are given in section 2. In section 3, we derive formulas for the distribution of an important random variable, and then in section 4, we apply these formulas to price a variable annuity with guaranteed minimum maturity benefit. Finally, we give some numerical results in section 5 and draw conclusion in section 6.

2. Details of the model and an important preliminary result

Suppose that $X = (X_t)_{t \geq 0}$ is a hyper-exponential jump diffusion process, i.e.,

$$X_t = X_0 + \mu t + \sigma W_t + \sum_{k=1}^{N_t} Z_k, \quad (2.1)$$

where $\sigma > 0$, μ and X_0 are constants; $\{W_t; t \geq 0\}$ is a standard Brownian motion; $\{\sum_{k=1}^{N_t} Z_k; t \geq 0\}$, independent of $\{W_t; t \geq 0\}$, is a compound Poisson process; the intensity of the Poisson process $\{N_t; t \geq 0\}$ is given by λ and the probability density function of Z_1 (denoted by $f_Z(z)$) is given by

$$f_Z(z) = \sum_{i=1}^m p_i \eta_i e^{-\eta_i z} \mathbf{1}_{\{z > 0\}} + \sum_{j=1}^n q_j \vartheta_j e^{\vartheta_j z} \mathbf{1}_{\{z < 0\}}. \quad (2.2)$$

Here, in (2.2), $\mathbf{1}_A$ is the indicator function of a set A ; $p_i > 0$, $\eta_i > 0$ for all $i = 1, \dots, m$; $q_j > 0$, $\vartheta_j > 0$ for all $j = 1, \dots, n$; $\sum_{i=1}^m p_i + \sum_{j=1}^n q_j = 1$. In addition, we assume that $\eta_1 < \eta_2 < \dots < \eta_m$ and $\vartheta_1 < \vartheta_2 < \dots < \vartheta_n$.

In this paper, we let S_t be the time- t value of one unit of the reference fund underlying a variable annuity and assume that

$$S_t = S_0 e^{X_t - X_0}. \quad (2.3)$$

Under the multi-layer expense strategy (1.2), if we denote by F_t the policyholder's account value at time t , then its dynamics are given by the following stochastic differential equation (SDE):

$$dF_t = F_{t-} \frac{dS_t}{S_{t-}} - \alpha_1 F_{t-} \mathbf{1}_{\{F_{t-} < B_1\}} dt - \alpha_2 F_{t-} \mathbf{1}_{\{F_{t-} \geq B_2\}} dt, \quad t > 0, \quad (2.4)$$

where α_i and B_i , $i = 1, 2$, are the deduction fee rate and the pre-specified level, respectively. Besides, the initial value F_0 is the single premium invested by the insured.

Remark 2.1. *Our approach can also be used to the following complicated fee structure:*

$$\int_0^t \left(\alpha_1 F_s \mathbf{1}_{\{F_s < B_1\}} + \sum_{i=2}^m \alpha_i F_s \mathbf{1}_{\{B_i \leq F_s < B_{i+1}\}} + \alpha_{m+1} F_s \mathbf{1}_{\{F_s \geq B_{m+2}\}} \right) ds, \quad (2.5)$$

where $B_1 \leq B_2 \leq \dots \leq B_{m+2}$. Note that formula (2.5) is similar to the so-called multi-layer dividend strategy, which is one of the motivations of this paper and has been studied by many papers (see Yang and Zhang (2009) for instance). This is the reason why our fee collecting method is called multi-layer expense strategy.

Set $U = (U_t)_{t \geq 0}$ be the unique strong solution to the following SDE

$$dU_t = dX_t - \alpha_1 \mathbf{1}_{\{U_t < b_1\}} dt - \alpha_2 \mathbf{1}_{\{U_t \geq b_2\}} dt, \quad t > 0, \quad (2.6)$$

and $U_0 = X_0$,

where $b_1 = \ln\left(\frac{B_1}{F_0}\right)$ and $b_2 = \ln\left(\frac{B_2}{F_0}\right)$. Then, it follows from Itô's formula that

$$F_t = F_0 e^{U_t}, \quad t \geq 0, \quad (2.7)$$

if $U_0 = 0$. In addition, by using a similar arguments to that in Remark 3 of Kyprianou and Loeffen (2010), one can conclude that the process U is a strong Markov process.

In this paper, we consider a VA with level GMMB, which has a payment $G(F_T)$ at its maturity T , where $G(\cdot)$ is a payoff function. An example of $G(\cdot)$ is that $G(x) = (K - x)_+$ for some $K > 0$. To price this VA, we need to compute the following expectation

$$E^* \left[e^{-rT} G(F_T) \right], \quad (2.8)$$

under an equivalent martingale measure P^* (i.e., $e^{-rt} e^{X_t}$ is martingale under P^*), where r denotes the continuously compounded constant risk-free rate.

Here, similar to Zhou and Wu (2015), we use the Cramér-Esscher transform (proposed in Gerber and Shiu (1994)) to obtain the pricing martingale measure P^* . An attractive property of the Cramér-Esscher transform is that the process X , under the martingale measure P^* , is also a hyper-exponential jump diffusion process (see, e.g., Appendix A in Asmussen et al. (2004)).

Therefore, without loss of generality, we suppose that the process X , under the martingale measure P^* , is given by (2.1) and (2.2). For the sake of brevity, in the following, we write simply P rather than P^* . Furthermore, we denote by \mathbb{P}_x the law of X such that $X_0 = x$ and by \mathbb{E}_x the corresponding expectation; and we write \mathbb{P} and \mathbb{E} when $X_0 = 0$ for simplicity. In addition, for $q > 0$, we let $e(q)$ be an exponential random variable with expectation $1/q$, which is independent of the process U under \mathbb{P}_x .

Of course, it is difficult to obtain the expression of (2.8) with F_t given by (2.7). But, if we can obtain the distribution of $U_{e(q)}$ for some $q > 0$, then the Laplace transform of (2.8) can be computed as follows:

$$\int_0^\infty e^{-sT} \mathbb{E} [e^{-rT} G(F_T)] dT = \frac{1}{s+r} \mathbb{E} [G(F_{e(s+r)})]. \quad (2.9)$$

Moreover, the total time of deducting fees, i.e.,

$$\int_0^T \mathbf{1}_{\{U_t < b_1\}} dt + \int_0^T \mathbf{1}_{\{U_t \geq b_2\}} dt, \quad (2.10)$$

can also be calculated from the distribution of $U_{e(q)}$ (see Remark 3.3). In short, deriving the expression for the probability distribution of $U_{e(q)}$ with $q > 0$ is a key task in this article.

Remark 2.2. *If we let τ represent the random variable denoting the time of death of the policyholder of a VA with level GMDB (whose payment at τ is $G_0(F_\tau)$), then we can compute the price of this VA as following:*

$$\mathbb{E} [e^{-r\tau} G_0(F_\tau) \mathbf{1}_{\{\tau < T\}}] = \int_0^T \mathbb{E} [e^{-rt} G_0(F_t)] g(t) dt, \quad (2.11)$$

where $g(t)$ is the density function of τ . As the term $\mathbb{E} [e^{-rt} G_0(F_t)]$ can be obtained from (2.9) via taking inverse Laplace transform, so (2.11) can be computed approximatively. However, this procedure involves massive numerical computations, especially when T is large. Alternatively, we can use a similar idea to that in Remark 2.3 of Zhou and Wu (2015) to calculate (2.11) by using the results on the distribution of $U_{e(q)}$.

The following lemma, which characters the probability distribution function of $U_{e(q)}$, gives us some important boundary conditions (see (2.14), (2.17) and (2.20)). The proof of Lemma 2.1 is very similar to that of Theorem 2.1 in Zhou and Wu (2015), thus we omit the details.

Lemma 2.1. (1) For given $y > b_2 > b_1$, consider a function $F_1(x)$ such that $F_1(x)$ is bounded and continuous on \mathbb{R} and twice continuously differentiable on \mathbb{R} except at b_1, b_2 and y . Assume $F_1(x)$ solves

$$\begin{aligned}\Gamma F_1(x) &= qF_1(x), \quad x < y \quad \text{and} \quad x \neq b_1, b_2, \\ \Gamma F_1(x) &= qF_1(x) - q, \quad x > y,\end{aligned}\tag{2.12}$$

where

$$\begin{aligned}\Gamma F_1(x) &= \frac{\sigma^2}{2} F_1''(x) + (\mu - \alpha_1 \mathbf{1}_{\{x < b_1\}} - \alpha_2 \mathbf{1}_{\{x \geq b_2\}}) F_1'(x) \\ &\quad + \lambda \int_{-\infty}^{\infty} F_1(x+z) f_Z(z) dz - \lambda F_1(x).\end{aligned}\tag{2.13}$$

Moreover, assume that the derivative of $F_1(x)$ is continuous at b_1, b_2 and y , i.e.,

$$F_1'(b_1-) = F_1'(b_1+), \quad F_1'(b_2-) = F_1'(b_2+) \quad \text{and} \quad F_1'(y-) = F_1'(y+).\tag{2.14}$$

Then

$$F_1(x) = \mathbb{P}_x(U_{e(q)} > y).\tag{2.15}$$

(2) For given $y < b_1 < b_2$, let $F_2(x)$ be bounded and continuous on \mathbb{R} and twice continuously differentiable on \mathbb{R} except at b_1, b_2 and y . If $F_2(x)$ solves

$$\begin{aligned}\Gamma F_2(x) &= qF_2(x), \quad x > y \quad \text{and} \quad x \neq b_1, b_2, \\ \Gamma F_2(x) &= qF_2(x) - q, \quad x < y,\end{aligned}\tag{2.16}$$

and satisfies

$$F_2'(b_1-) = F_2'(b_1+), \quad F_2'(b_2-) = F_2'(b_2+) \quad \text{and} \quad F_2'(y-) = F_2'(y+),\tag{2.17}$$

then

$$F_2(x) = \mathbb{P}_x(U_{e(q)} < y).\tag{2.18}$$

(3) For given $b_1 < y < b_2$, let $F_3(x)$ be bounded and continuous on \mathbb{R} and twice continuously differentiable on \mathbb{R} except at b_1, b_2 and y . If $F_3(x)$ solves

$$\begin{aligned}\Gamma F_3(x) &= qF_3(x), \quad x > b_1 \quad \text{and} \quad b_1 < x < y, \\ \Gamma F_3(x) &= qF_3(x) - q, \quad x > y \quad \text{and} \quad x \neq b_2,\end{aligned}\tag{2.19}$$

and satisfies

$$F_3'(b_1-) = F_3'(b_1+), \quad F_3'(b_2-) = F_3'(b_2+) \quad \text{and} \quad F_3'(y-) = F_3'(y+),\tag{2.20}$$

then

$$F_3(x) = \mathbb{P}_x(U_{e(q)} > y).\tag{2.21}$$

□

Remark 2.3. Although Lemma 2.1 holds, it is not easy to obtain the distribution of $U_{e(q)}$ by solving (2.12), (2.16) and (2.19) directly. In next section, we derive the expression of the probability distribution of $U_{e(q)}$ by using an another approach rather than solving the equations in Lemma 2.1.

3. The distribution of $U_{e(q)}$

In this section, for the process U determined by (2.1), (2.2) and (2.6), we first consider the case that $b_1 < b_2$ and derive formulas for $\mathbb{P}_x(U_{e(q)} > y)$ with $y > b_2$, $\mathbb{P}_x(U_{e(q)} < y)$ with $y < b_1$ and $\mathbb{P}_x(U_{e(q)} > y)$ with $b_1 < y < b_2$. After that the corresponding results for the case of $b_1 = b_2$ are obtained by letting $b_2 \downarrow b_1$. Before presenting the results, we introduce some notation to end this paragraph. The Lévy exponent of the process X in (2.1) is given by

$$\psi(z) := \ln(\mathbb{E}[e^{zX_1}]) = \frac{\sigma^2}{2}z^2 + \mu z + \lambda \left(\sum_{i=1}^m \frac{p_i \eta_i}{\eta_i - z} + \sum_{j=1}^n \frac{q_j \vartheta_j}{\vartheta_j + z} - 1 \right). \quad (3.1)$$

For given $q > 0$, the equation $\psi(z) = q$ has exactly $(m + n + 2)$ real roots (denoted by $\beta_1, \beta_2, \dots, \beta_{m+1}, -\gamma_1, -\gamma_2, \dots, -\gamma_{n+1}$), which satisfy (see Lemma 2.1 in Cai (2009) for the proof)

$$\begin{aligned} 0 < \beta_1 < \eta_1 < \beta_2 < \dots < \eta_m < \beta_{m+1} < \infty, \\ 0 < \gamma_1 < \vartheta_1 < \gamma_2 < \dots < \vartheta_n < \gamma_{n+1} < \infty. \end{aligned} \quad (3.2)$$

For the following two equations:

$$\tilde{\psi}(z) := \psi(z) - \alpha_1 z = q \quad \text{and} \quad \hat{\psi}(z) := \psi(z) - \alpha_2 z = q, \quad (3.3)$$

the corresponding roots are denoted respectively by $\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_{m+1}, -\tilde{\gamma}_1, -\tilde{\gamma}_2, \dots, -\tilde{\gamma}_{n+1}$ and $\hat{\beta}_1, \hat{\beta}_2, \dots, \hat{\beta}_{m+1}, -\hat{\gamma}_1, -\hat{\gamma}_2, \dots, -\hat{\gamma}_{n+1}$.

The following three theorems give the expressions for $\mathbb{P}_x(U_{e(q)} > y)$ with $y > b_2$, $\mathbb{P}_x(U_{e(q)} < y)$ with $y < b_1$ and $\mathbb{P}_x(U_{e(q)} > y)$ with $b_1 < y < b_2$, respectively. We give the details of the proof of Theorem 3.1 in Appendix B and omit the proofs of Theorems 3.2 and 3.3 as they are similar.

Theorem 3.1. *For given $b_1 < b_2$, α_1 and α_2 in (2.6), the expression of $\mathbb{P}_x(U_{e(q)} > y)$ for $y > b_2$ is given as follows.*

$$\mathbb{P}_x(U_{e(q)} > y) = \begin{cases} \sum_{i=1}^{m+1} E_i e^{\tilde{\beta}_i(x-b_1)}, & x \leq b_1, \\ \sum_{i=1}^{m+1} F_i e^{\beta_i(x-b_2)} + \sum_{j=1}^{n+1} G_j e^{\gamma_j(b_1-x)}, & b_1 \leq x \leq b_2, \\ \sum_{i=1}^{m+1} H_i e^{\tilde{\beta}_i(x-y)} + \sum_{j=1}^{n+1} M_j e^{\hat{\gamma}_j(b_2-x)}, & b_2 \leq x \leq y, \\ 1 + \sum_{j=1}^{n+1} N_j e^{\hat{\gamma}_j(y-x)} + \sum_{j=1}^{n+1} M_j e^{\hat{\gamma}_j(b_2-x)}, & x \geq y, \end{cases} \quad (3.4)$$

where

$$\begin{aligned} H_i &= \frac{\prod_{k=1}^m (\eta_k - \hat{\beta}_i) \prod_{k=1, k \neq i}^{m+1} \hat{\beta}_k \prod_{k=1}^m (\hat{\beta}_i + \vartheta_k) \prod_{k=1}^{m+1} \hat{\gamma}_k}{\prod_{k=1}^m \eta_k \prod_{k=1, k \neq i}^{m+1} (\hat{\beta}_k - \hat{\beta}_i) \prod_{k=1}^m \vartheta_k \prod_{k=1}^{m+1} (\hat{\beta}_i + \hat{\gamma}_k)}, \quad 1 \leq i \leq m+1, \\ N_j &= -\frac{\prod_{k=1}^n (\vartheta_k - \hat{\gamma}_j) \prod_{k=1, k \neq j}^{n+1} \hat{\gamma}_k \prod_{k=1}^m (\hat{\gamma}_j + \eta_k) \prod_{k=1}^{m+1} \hat{\beta}_k}{\prod_{k=1}^n \vartheta_k \prod_{k=1, k \neq j}^{n+1} (\hat{\gamma}_k - \hat{\gamma}_j) \prod_{k=1}^m \eta_k \prod_{k=1}^{m+1} (\hat{\gamma}_j + \hat{\beta}_k)}, \quad 1 \leq j \leq n+1. \end{aligned} \quad (3.5)$$

The other constants in (3.4), E_1, \dots, E_{m+1} , F_1, \dots, F_{m+1} , G_1, \dots, G_{n+1} and M_1, \dots, M_{n+1} , are determined by

$$(E_1, \dots, E_{m+1}, F_1, \dots, F_{m+1}, G_1, \dots, G_{n+1}, M_1, \dots, M_{n+1}) Q_1 = h, \quad (3.6)$$

with Q_1 given by (A.1) in Appendix A and

$$h = (0, \dots, 0, h_{m+n+3}, \dots, h_{2m+2n+4}), \quad (3.7)$$

where

$$\begin{aligned} h_{m+n+3} &= \sum_{i=1}^{m+1} H_i e^{\hat{\beta}_i(b_2-y)}, \quad h_{m+n+4} = \sum_{i=1}^{m+1} H_i \hat{\beta}_i e^{\hat{\beta}_i(b_2-y)}, \\ h_{m+n+4+k} &= \sum_{i=1}^{m+1} \frac{H_i \vartheta_k}{\vartheta_k + \hat{\beta}_i} e^{\hat{\beta}_i(b_2-y)}, \quad k = 1, \dots, n, \\ h_{m+2n+4+k} &= \sum_{i=1}^{m+1} \frac{H_i \eta_k}{\eta_k - \hat{\beta}_i} e^{\hat{\beta}_i(b_2-y)}, \quad k = 1, \dots, m. \end{aligned} \quad (3.8)$$

Remark 3.1. Intuitively, the columns of the matrix Q_1 in (A.1) are linearly independent. In other words, the matrix Q_1 is nonsingular. At present, we cannot find an easy approach to show this fact. However, a large number of numerical calculations, including those in section 5, confirm that Q_1 is an invertible matrix.

Theorem 3.2. For given $b_1 < b_2$, α_1 and α_2 in (2.6), the expression of $\mathbb{P}_x(U_{e(q)} < y)$ for $y < b_1$ is given as follows.

$$\mathbb{P}_x(U_{e(q)} < y) = \begin{cases} 1 + \sum_{i=1}^{m+1} \tilde{E}_i e^{\tilde{\beta}_i(x-y)} + \sum_{i=1}^{m+1} \tilde{F}_i e^{\tilde{\beta}_i(x-b_1)}, & x \leq y, \\ \sum_{i=1}^{m+1} \tilde{F}_i e^{\tilde{\beta}_i(x-b_1)} + \sum_{j=1}^{n+1} \tilde{G}_j e^{\tilde{\gamma}_j(y-x)}, & y \leq x \leq b_1, \\ \sum_{j=1}^{n+1} \tilde{H}_j e^{\tilde{\gamma}_j(b_1-x)} + \sum_{i=1}^{m+1} \tilde{M}_i e^{\beta_i(x-b_2)}, & b_1 \leq x \leq b_2, \\ \sum_{j=1}^{n+1} \tilde{N}_j e^{\tilde{\gamma}_j(b_2-x)}, & x \geq b_2, \end{cases} \quad (3.9)$$

where

$$\begin{aligned}\tilde{G}_j &= \frac{\prod_{k=1}^n (\vartheta_k - \tilde{\gamma}_j) \prod_{k=1, k \neq j}^{n+1} \tilde{\gamma}_k \prod_{k=1}^m (\tilde{\gamma}_j + \eta_k) \prod_{k=1}^{m+1} \tilde{\beta}_k}{\prod_{k=1}^n \vartheta_k \prod_{k=1, k \neq j}^{n+1} (\tilde{\gamma}_k - \tilde{\gamma}_j) \prod_{k=1}^m \eta_k \prod_{k=1}^{m+1} (\tilde{\gamma}_j + \tilde{\beta}_k)}, \quad 1 \leq j \leq n+1, \\ \tilde{E}_i &= -\frac{\prod_{k=1}^m (\eta_k - \tilde{\beta}_i) \prod_{k=1, k \neq i}^{m+1} \tilde{\beta}_k \prod_{k=1}^n (\tilde{\beta}_i + \vartheta_k) \prod_{k=1}^{n+1} \tilde{\gamma}_k}{\prod_{k=1}^m \eta_k \prod_{k=1, k \neq i}^{m+1} (\tilde{\beta}_k - \tilde{\beta}_i) \prod_{k=1}^n \vartheta_k \prod_{k=1}^{n+1} (\tilde{\beta}_i + \tilde{\gamma}_k)}, \quad 1 \leq i \leq m+1,\end{aligned}\tag{3.10}$$

and

$$\left(\tilde{F}_1, \dots, \tilde{F}_{m+1}, \tilde{M}_1, \dots, \tilde{M}_{m+1}, \tilde{H}_1, \dots, \tilde{H}_{n+1}, \tilde{N}_1, \dots, \tilde{N}_{n+1} \right) Q_1 = \tilde{h}, \tag{3.11}$$

with

$$\tilde{h} = \left(\tilde{h}_1, \dots, \tilde{h}_{2+n+m}, 0, \dots, 0 \right), \tag{3.12}$$

and

$$\begin{aligned}\tilde{h}_1 &= -\sum_{j=1}^{n+1} \tilde{G}_j e^{\tilde{\gamma}_j(y-b_1)}, \quad \tilde{h}_2 = \sum_{j=1}^{n+1} \tilde{G}_j \tilde{\gamma}_j e^{\tilde{\gamma}_j(y-b_1)}, \\ \tilde{h}_{2+k} &= -\sum_{j=1}^{n+1} \frac{\tilde{G}_j \vartheta_k}{\vartheta_k - \tilde{\gamma}_j} e^{\tilde{\gamma}_j(y-b_1)}, \quad k = 1, \dots, n, \\ \tilde{h}_{2+n+k} &= -\sum_{j=1}^{n+1} \frac{\tilde{G}_j \eta_k}{\eta_k + \tilde{\gamma}_j} e^{\tilde{\gamma}_j(y-b_1)}, \quad k = 1, \dots, m.\end{aligned}\tag{3.13}$$

Theorem 3.3. For given $b_1 < y < b_2$, α_1 and α_2 in (2.6), we have the following results.

$$\mathbb{P}_x(U_{e(q)} > y) = \begin{cases} \sum_{i=1}^{m+1} \hat{E}_i e^{\tilde{\beta}_i(x-b_1)}, & x \leq b_1, \\ \sum_{i=1}^{m+1} \left(\hat{U}_i + \hat{H}_i e^{\beta_i(y-b_2)} \right) e^{\beta_i(x-y)} + \sum_{j=1}^{n+1} \hat{G}_j e^{\gamma_j(b_1-x)}, & b_1 \leq x \leq y, \\ 1 + \sum_{i=1}^{m+1} \hat{H}_i e^{\beta_i(x-b_2)} + \sum_{j=1}^{n+1} \left(\hat{V}_j + \hat{G}_j e^{\gamma_j(b_1-y)} \right) e^{\gamma_j(y-x)}, & y \leq x \leq b_2, \\ 1 + \sum_{j=1}^{n+1} \hat{N}_j e^{\tilde{\gamma}_j(b_2-x)}, & x \geq b_2, \end{cases}\tag{3.14}$$

where

$$\begin{aligned}\hat{V}_j &= -\frac{\prod_{k=1}^n (\vartheta_k - \gamma_j) \prod_{k=1, k \neq j}^{n+1} \gamma_k \prod_{k=1}^m (\gamma_j + \eta_k) \prod_{k=1}^{m+1} \beta_k}{\prod_{k=1}^n \vartheta_k \prod_{k=1, k \neq j}^{n+1} (\gamma_k - \gamma_j) \prod_{k=1}^m \eta_k \prod_{k=1}^{m+1} (\gamma_j + \beta_k)}, \quad 1 \leq j \leq n+1, \\ \hat{U}_i &= \frac{\prod_{k=1}^m (\eta_k - \beta_i) \prod_{k=1, k \neq i}^{m+1} \beta_k \prod_{k=1}^n (\beta_i + \vartheta_k) \prod_{k=1}^{n+1} \gamma_k}{\prod_{k=1}^m \eta_k \prod_{k=1, k \neq i}^{m+1} (\beta_k - \beta_i) \prod_{k=1}^n \vartheta_k \prod_{k=1}^{n+1} (\beta_i + \gamma_k)}, \quad 1 \leq i \leq m+1,\end{aligned}\tag{3.15}$$

and

$$\left(\hat{E}_1, \dots, \hat{E}_{m+1}, \hat{H}_1, \dots, \hat{H}_{m+1}, \hat{G}_1, \dots, \hat{G}_{n+1}, \hat{N}_1, \dots, \hat{N}_{n+1} \right) Q_1 = \hat{h}. \quad (3.16)$$

Here, the vector \hat{h} in (3.16) is given by

$$\hat{h} = \left(\hat{h}_1, \dots, \hat{h}_{2m+2n+4} \right), \quad (3.17)$$

where

$$\begin{aligned} \hat{h}_1 &= \sum_{i=1}^{m+1} \hat{U}_i e^{\beta_i(b_1-y)}, \quad \hat{h}_2 = \sum_{i=1}^{m+1} \hat{U}_i \beta_i e^{\beta_i(b_1-y)}, \\ \hat{h}_{2+k} &= \sum_{i=1}^{m+1} \frac{\hat{U}_i \vartheta_k}{\vartheta_k + \beta_i} e^{\beta_i(b_1-y)}, \quad k = 1, \dots, n, \\ \hat{h}_{2+n+k} &= \sum_{i=1}^{m+1} \frac{\hat{U}_i \eta_k}{\eta_k - \beta_i} e^{\beta_i(b_1-y)}, \quad k = 1, \dots, m, \end{aligned} \quad (3.18)$$

and

$$\begin{aligned} \hat{h}_{m+n+3} &= - \sum_{j=1}^{n+1} \hat{V}_j e^{\gamma_j(y-b_2)}, \quad \hat{h}_{m+n+4} = \sum_{j=1}^{n+1} \hat{V}_j \gamma_j e^{\gamma_j(y-b_2)}, \\ \hat{h}_{m+n+4+k} &= - \sum_{j=1}^{n+1} \frac{\hat{V}_j \vartheta_k}{\vartheta_k - \gamma_j} e^{\gamma_j(y-b_2)}, \quad k = 1, \dots, n, \\ \tilde{h}_{m+2n+4+k} &= - \sum_{j=1}^{n+1} \frac{\hat{V}_j \eta_k}{\eta_k + \gamma_j} e^{\gamma_j(y-b_2)}, \quad k = 1, \dots, m. \end{aligned} \quad (3.19)$$

Remark 3.2. From the derivation of Theorem 3.3, we have

$$\begin{aligned} \sum_{i=1}^{m+1} \hat{U}_i - \sum_{j=1}^{n+1} \hat{V}_j - 1 &= 0, \\ \sum_{i=1}^{m+1} \hat{U}_i \beta_i + \sum_{j=1}^{n+1} \hat{V}_j \gamma_j &= 0, \\ \sum_{i=1}^{m+1} \frac{\hat{U}_i \vartheta_k}{\vartheta_k + \beta_i} - \sum_{j=1}^{n+1} \frac{\hat{V}_j \vartheta_k}{\vartheta_k - \gamma_j} - 1 &= 0, \quad k = 1, 2, \dots, n, \\ \sum_{i=1}^{m+1} \frac{\hat{U}_i \eta_k}{\eta_k - \beta_i} - \sum_{j=1}^{n+1} \frac{\hat{V}_j \eta_k}{\eta_k + \gamma_j} - 1 &= 0, \quad k = 1, 2, \dots, m. \end{aligned} \quad (3.20)$$

We remark that formula (3.15) is obtained easily via solving (3.20) and that the equations in (3.20) will be used in proof of the following Corollary 3.1.

In order to obtain the distribution of $U_{e(q)}$, the above three theorems are not enough as they do not give the values of the two probabilities: $\mathbb{P}_x(U_{e(q)} = b_2)$ and $\mathbb{P}_x(U_{e(q)} = b_1)$. In Corollary 3.1 (whose proof is given in Appendix C), we draw the conclusion that both of them are equal to zero. We remark that this result is not surprising because we have assumed that $\sigma > 0$ in this paper.

Corollary 3.1. *For given $b_1 < b_2$, α_1 and α_2 in (2.6), we have*

$$\mathbb{P}_x(U_{e(q)} = b_2) = \mathbb{P}_x(U_{e(q)} = b_1) = 0. \quad (3.21)$$

Remark 3.3. *For $y \in \mathbb{R}$, applying integration by part yields*

$$\begin{aligned} \int_0^\infty e^{-qT} \mathbb{E}_x \left[\int_0^T \mathbf{1}_{\{U_t \geq y\}} dt \right] dT &= \frac{\mathbb{P}_x(U_{e(q)} \geq y)}{q^2}, \\ \int_0^\infty e^{-qT} \mathbb{E}_x \left[\int_0^T \mathbf{1}_{\{U_t < y\}} dt \right] dT &= \frac{\mathbb{P}_x(U_{e(q)} < y)}{q^2}, \end{aligned} \quad (3.22)$$

which can be computed from Theorems 3.1, 3.2, 3.3 and Corollary 3.1. Particularly, note that the sum of $\mathbb{E}_x \left[\int_0^T \mathbf{1}_{\{U_t \geq b_2\}} dt \right]$ and $\mathbb{E}_x \left[\int_0^T \mathbf{1}_{\{U_t < b_1\}} dt \right]$ is the total time of deducting fees for a VA with GMMB rider under the multi-layer expense strategy (1.2).

For fixed b_1 , if we let $b_2 \downarrow b_1$ in Theorems 3.1 and 3.2, then we can derive formulas for the distribution function of $U_{e(q)}$ with $b_1 = b_2$ in the following Corollary 3.2. In Appendix C, we give the details of deriving Corollary 3.2.

Corollary 3.2. *For given $b_1 = b_2$, α_1 and α_2 in (2.6), we have the following results.*

(i) *For $y > b_1$,*

$$\mathbb{P}_x(U_{e(q)} > y) = \begin{cases} \sum_{i=1}^{m+1} E_i^1 e^{\tilde{\beta}_i(x-b_1)}, & x \leq b_1, \\ \sum_{i=1}^{m+1} H_i e^{\hat{\beta}_i(x-y)} + \sum_{j=1}^{n+1} M_j^1 e^{\hat{\gamma}_j(b_1-x)}, & b_1 \leq x \leq y, \\ 1 + \sum_{j=1}^{n+1} N_j e^{\hat{\gamma}_j(y-x)} + \sum_{j=1}^{n+1} M_j^1 e^{\hat{\gamma}_j(b_1-x)}, & x \geq y, \end{cases} \quad (3.23)$$

where H_i and N_j are given by (3.5) and

$$\begin{aligned} M_j^1 &= \frac{\prod_{k=1}^m (\hat{\gamma}_j + \eta_k) \prod_{k=1}^n (\vartheta_k - \hat{\gamma}_j) \prod_{k=1}^{m+1} \hat{\beta}_k \prod_{k=1}^{n+1} \hat{\gamma}_k}{\prod_{k=1}^m \eta_k \prod_{k=1}^n \vartheta_k \prod_{k=1}^{m+1} (\hat{\gamma}_j + \tilde{\beta}_k) \prod_{k=1, k \neq j}^{n+1} (\hat{\gamma}_k - \hat{\gamma}_j)} \\ &\times \sum_{i=1}^{m+1} \frac{\prod_{k=1}^{m+1} (\hat{\beta}_i - \tilde{\beta}_k) e^{\hat{\beta}_i(b_1-y)}}{\hat{\beta}_i (\hat{\beta}_i + \hat{\gamma}_j) \prod_{k=1, k \neq i}^{m+1} (\hat{\beta}_i - \tilde{\beta}_k)}, \quad j = 1, \dots, n+1, \end{aligned} \quad (3.24)$$

and

$$E_i^1 = \frac{\prod_{k=1}^{n+1} \hat{\beta}_k \prod_{k=1}^{n+1} \hat{\gamma}_k \prod_{k=1}^m (\tilde{\beta}_i - \eta_k) \prod_{k=1}^n (\tilde{\beta}_i + \vartheta_k)}{\prod_{k=1}^m \eta_k \prod_{k=1}^n \vartheta_k \prod_{k=1, k \neq i}^{m+1} (\tilde{\beta}_i - \tilde{\beta}_k) \prod_{k=1}^{n+1} (\tilde{\beta}_i + \hat{\gamma}_k)} \times \sum_{j=1}^{m+1} \frac{\prod_{k=1, k \neq i}^{m+1} (\hat{\beta}_j - \tilde{\beta}_k)}{\hat{\beta}_j \prod_{k=1, k \neq j}^{m+1} (\hat{\beta}_j - \hat{\beta}_k)} e^{\hat{\beta}_j(b_1 - y)}, \quad i = 1, \dots, m+1. \quad (3.25)$$

(ii) For $y < b_1$,

$$\mathbb{P}_x(U_{e(q)} < y) = \begin{cases} 1 + \sum_{i=1}^{m+1} \tilde{E}_i e^{\tilde{\beta}_i(x-y)} + \sum_{i=1}^{m+1} \tilde{F}_i^1 e^{\tilde{\beta}_i(x-b_1)}, & x \leq y, \\ \sum_{i=1}^{m+1} \tilde{F}_i^1 e^{\tilde{\beta}_i(x-b_1)} + \sum_{j=1}^{n+1} \tilde{G}_j e^{\tilde{\gamma}_j(y-x)}, & y \leq x \leq b_1, \\ \sum_{j=1}^{n+1} \tilde{N}_j^1 e^{\tilde{\gamma}_j(b_1-x)}, & x \geq b_1, \end{cases} \quad (3.26)$$

where \tilde{E}_i and \tilde{G}_j are given by (3.10) and

$$\tilde{N}_j^1 = \frac{\prod_{k=1}^{m+1} \tilde{\beta}_k \prod_{k=1}^{n+1} \tilde{\gamma}_k \prod_{k=1}^m (\hat{\gamma}_j + \eta_k) \prod_{k=1}^n (\vartheta_k - \hat{\gamma}_j)}{\prod_{k=1}^m \eta_k \prod_{k=1}^n \vartheta_k \prod_{k=1}^{m+1} (\hat{\gamma}_j + \tilde{\beta}_k) \prod_{k=1, k \neq j}^{n+1} (\hat{\gamma}_k - \hat{\gamma}_j)} \times \sum_{i=1}^{n+1} \frac{\prod_{k=1, k \neq j}^{n+1} (\hat{\gamma}_k - \tilde{\gamma}_i)}{\tilde{\gamma}_i \prod_{k=1, k \neq i}^{n+1} (\tilde{\gamma}_k - \tilde{\gamma}_i)} e^{\tilde{\gamma}_i(y-b_1)}, \quad j = 1, \dots, n+1, \quad (3.27)$$

$$\tilde{F}_i^1 = \frac{\prod_{k=1}^{m+1} \tilde{\beta}_k \prod_{k=1}^{n+1} \tilde{\gamma}_k \prod_{k=1}^m (\tilde{\beta}_i - \eta_k) \prod_{k=1}^n (\tilde{\beta}_i + \vartheta_k)}{\prod_{k=1}^m \eta_k \prod_{k=1}^n \vartheta_k \prod_{k=1, k \neq i}^{m+1} (\tilde{\beta}_i - \tilde{\beta}_k) \prod_{k=1}^{n+1} (\tilde{\beta}_i + \hat{\gamma}_k)} \times \sum_{j=1}^{n+1} \frac{\prod_{k=1}^{n+1} (\hat{\gamma}_k - \tilde{\gamma}_j) e^{\tilde{\gamma}_j(y-b_1)}}{-\tilde{\gamma}_j (\tilde{\beta}_i + \tilde{\gamma}_j) \prod_{k=1, k \neq j}^{n+1} (\tilde{\gamma}_k - \tilde{\gamma}_j)}, \quad i = 1, \dots, m+1. \quad (3.28)$$

(iii) For $x \in \mathbb{R}$,

$$\mathbb{P}_x(U_{e(q)} = b_1) = 0. \quad (3.29)$$

Remark 3.4. Corollary 3.2 extends the results in Theorem 3.1 of Zhou and Wu (2015) from the double exponential jump diffusion process to the hyper-exponential jump diffusion process.

4. Evaluating variable annuities under the multi-layer expense strategy

In this section, we apply the results in section 3 to evaluate a variable annuity with the multi-layer expense strategy. For the sake of simplicity, we only

investigate the case that $G(x)$ in (2.8) is given by $G(x) = (K - x)_+$ for some guaranteed level K .

For given $B_1 \leq B_2$ in (2.4), the fair fee rates α_1^* and α_2^* are computed such that the initial premium equals the expected value of the discounted payoff, i.e.,

$$F_0 = \mathbb{E} [e^{-rT} \max\{F_T, K\}] = \mathbb{E} [e^{-rT} F_T] + \mathbb{E} [e^{-rT} (K - F_T)_+]. \quad (4.1)$$

Besides, with the fair fee rates α_1^* and α_2^* , we want to calculate the total fees that will be deducted, i.e.,

$$\int_0^T e^{-rt} (\alpha_1^* F_t \mathbf{1}_{\{F_t < B_1\}} + \alpha_2^* F_t \mathbf{1}_{\{F_t \geq B_2\}}) dt, \quad (4.2)$$

whose expectation equals (by Itô's formula)

$$\begin{aligned} & \mathbb{E} \left[\int_0^T e^{-rt} (\alpha_1^* F_t \mathbf{1}_{\{F_t < B_1\}} + \alpha_2^* F_t \mathbf{1}_{\{F_t \geq B_2\}}) dt \right] \\ &= \mathbb{E} [e^{-rT} (F_0 e^{X_T} - F_T)] = F_0 - \mathbb{E} [e^{-rT} F_T], \end{aligned} \quad (4.3)$$

where the second equality follows from the fact that $e^{-rt} e^{X_t}$ is a martingale.

Formulas (4.1) and (4.3) imply that we need to calculate the two expectations: $\mathbb{E} [e^{-rT} F_T]$ and $\mathbb{E} [e^{-rT} (K - F_T)_+]$ with given B_1 , B_2 , α_1 and α_2 . In the following, we will derive the Laplace transforms of these two expectations with respect to T . In order to avoid introducing more notation, we only consider the case that $K = F_0$, $B_1 = F_0 (= K)$ and $B_2 > F_0$ (other cases can be discussed in a similar way).

Remark 4.1. *The case that K equals F_0 is known as the "return-of-premium" guarantee. In addition, the situation of $B_1 = F_0 = K$ is interesting, as it means that the insurer deducts fees for the embedded guarantee if the guarantee (like a put option) is in-the-money.*

- Results on $\int_0^\infty e^{-sT} \mathbb{E} [e^{-rT} F_T] dT = \frac{1}{q} \mathbb{E} [F_{e(q)}]$ with $q = r + s$.

For $b_1 = \ln(B_1/F_0) = 0$ and $b_2 = \ln(B_2/F_0) > 0$, we have

$$\begin{aligned} \mathbb{E} [F_{e(q)}] &= \int_{-\infty}^\infty F_0 e^y \mathbb{P} (U_{e(q)} \in dy) = F_0 - \int_{-\infty}^0 F_0 e^y \mathbb{P} (U_{e(q)} < y) dy \\ &\quad + \int_0^{b_2} F_0 e^y \mathbb{P} (U_{e(q)} > y) dy + \int_{b_2}^\infty F_0 e^y \mathbb{P} (U_{e(q)} > y) dy, \end{aligned} \quad (4.4)$$

where we have used the fact that $\lim_{y \uparrow \infty} e^y \mathbb{P} (U_{e(q)} > y) = 0$ (which is due to that $\mathbb{E} [e^{U_{e(q)}}] < \mathbb{E} [e^{X_{e(q)}}] < \infty$ for $q > r$) in the second equality.

It follows from Theorem 3.1 that

$$\begin{aligned} & \int_{b_2}^\infty e^y \mathbb{P} (U_{e(q)} > y) dy = \int_{b_2}^\infty e^y \sum_{i=1}^{m+1} E_i dy \\ &= \left(\int_{b_2}^\infty e^y h_1 dy, \dots, \int_{b_2}^\infty e^y h_{2m+2n+4} dy \right) Q_1^{-1} w^T, \end{aligned} \quad (4.5)$$

where

$$w = (\underbrace{1, \dots, 1}_{m+1}, \underbrace{0, \dots, 0}_{m+2n+3}). \quad (4.6)$$

Similarly, from Theorems 3.2 and 3.3, we have

$$\begin{aligned} \int_{-\infty}^0 e^y \mathbb{P}(U_{e(q)} < y) dy &= \int_{-\infty}^0 e^y \left(\sum_{i=1}^{m+1} \tilde{F}_i + \sum_{j=1}^{n+1} \tilde{G}_j e^{\tilde{\gamma}_j y} \right) dy \\ &= \sum_{j=1}^{n+1} \frac{\tilde{G}_j}{1 + \tilde{\gamma}_j} + \left(\int_{-\infty}^0 e^y \tilde{h}_1 dy, \dots, \int_{-\infty}^0 e^y \tilde{h}_{2m+2n+4} dy \right) Q_1^{-1} w^T, \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} \int_0^{b_2} e^y \mathbb{P}(U_{e(q)} > y) dy &= \int_0^{b_2} e^y \sum_{i=1}^{m+1} \hat{E}_i dy \\ &= \left(\int_0^{b_2} e^y \hat{h}_1 dy, \dots, \int_0^{b_2} e^y \hat{h}_{2m+2n+4} dy \right) Q_1^{-1} w^T. \end{aligned} \quad (4.8)$$

Therefore, we obtain that

$$\frac{1}{q} \mathbb{E} [F_{e(q)}] = \frac{F_0}{q} + \frac{1}{q} F_0(v_1, \dots, v_{2m+2n+4}) Q_1^{-1} w^T - \frac{1}{q} \sum_{j=1}^{n+1} \frac{F_0 \tilde{G}_j}{1 + \tilde{\gamma}_j}, \quad (4.9)$$

where

$$\begin{aligned} v_1 &= \sum_{i=1}^{m+1} \frac{\hat{U}_i}{\beta_i - 1} \left(1 - e^{(1-\beta_i)b_2} \right) + \sum_{j=1}^{n+1} \frac{\tilde{G}_j}{1 + \tilde{\gamma}_j}, \\ v_2 &= \sum_{i=1}^{m+1} \frac{\hat{U}_i \beta_i}{\beta_i - 1} \left(1 - e^{(1-\beta_i)b_2} \right) - \sum_{j=1}^{n+1} \frac{\tilde{G}_j \tilde{\gamma}_j}{1 + \tilde{\gamma}_j}, \\ v_{2+k} &= \sum_{i=1}^{m+1} \frac{\hat{U}_i \vartheta_k \left(1 - e^{(1-\beta_i)b_2} \right)}{(\vartheta_k + \beta_i)(\beta_i - 1)} + \sum_{j=1}^{n+1} \frac{\tilde{G}_j \vartheta_k}{(\vartheta_k - \tilde{\gamma}_j)(1 + \tilde{\gamma}_j)}, \quad k = 1, \dots, n, \\ v_{2+n+k} &= \sum_{i=1}^{m+1} \frac{\hat{U}_i \eta_k \left(1 - e^{(1-\beta_i)b_2} \right)}{(\eta_k - \beta_i)(\beta_i - 1)} + \sum_{j=1}^{n+1} \frac{\tilde{G}_j \eta_k}{(\eta_k + \tilde{\gamma}_j)(1 + \tilde{\gamma}_j)}, \quad k = 1, \dots, m, \end{aligned} \quad (4.10)$$

and

$$\begin{aligned}
v_{m+n+3} &= \sum_{j=1}^{n+1} \frac{\hat{V}_j}{\gamma_j + 1} (e^{-\gamma_j b_2} - e^{b_2}) + \sum_{i=1}^{m+1} \frac{\tilde{H}_i e^{b_2}}{\hat{\beta}_i - 1}, \\
v_{m+n+4} &= \sum_{j=1}^{n+1} \frac{\hat{V}_j \gamma_j}{\gamma_j + 1} (e^{b_2} - e^{-\gamma_j b_2}) + \sum_{i=1}^{m+1} \frac{H_i \hat{\beta}_i e^{b_2}}{\hat{\beta}_i - 1}, \\
v_{m+n+4+k} &= \sum_{j=1}^{n+1} \frac{\hat{V}_j \vartheta_k (e^{-\gamma_j b_2} - e^{b_2})}{(\vartheta_k - \gamma_j)(\gamma_j + 1)} + \sum_{i=1}^{m+1} \frac{H_i \vartheta_k e^{b_2}}{(\vartheta_k + \hat{\beta}_i)(\hat{\beta}_i - 1)}, \quad 1 \leq k \leq n, \\
v_{m+2n+4+k} &= \sum_{j=1}^{n+1} \frac{\hat{V}_j \eta_k (e^{-\gamma_j b_2} - e^{b_2})}{(\eta_k + \gamma_j)(\gamma_j + 1)} + \sum_{i=1}^{m+1} \frac{H_i \eta_k e^{b_2}}{(\eta_k - \hat{\beta}_i)(\hat{\beta}_i - 1)}, \quad 1 \leq k \leq m.
\end{aligned} \tag{4.11}$$

- Results on $\int_0^\infty e^{-sT} \mathbb{E} [e^{-rT} (F_0 - F_T)_+] dT = \frac{1}{q} \mathbb{E} [(F_0 - F_{e(q)})_+]$.
For $b_1 = \ln(B_1/F_0) = 0$ and $b_2 = \ln(B_2/F_0) > 0$, we have

$$\begin{aligned}
\frac{1}{q} \mathbb{E} [(F_0 - F_{e(q)})_+] &= \frac{F_0}{q} \int_{-\infty}^0 e^y \mathbb{P}(U_{e(q)} < y) dy \\
&= \frac{1}{q} \sum_{j=1}^{n+1} \frac{F_0 \tilde{G}_j}{1 + \tilde{\gamma}_j} + \frac{1}{q} F_0 (\tilde{v}_1, \dots, \tilde{v}_{2m+2n+4}) Q_1^{-1} w^T,
\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}
\tilde{v}_1 &= - \sum_{j=1}^{n+1} \frac{\tilde{G}_j}{1 + \tilde{\gamma}_j}, \quad \tilde{v}_2 = \sum_{j=1}^{n+1} \frac{\tilde{G}_j \tilde{\gamma}_j}{1 + \tilde{\gamma}_j}, \\
\tilde{v}_{2+k} &= - \sum_{j=1}^{n+1} \frac{\tilde{G}_j \vartheta_k}{(\vartheta_k - \tilde{\gamma}_j)(1 + \tilde{\gamma}_j)}, \quad k = 1, 2, \dots, n, \\
\tilde{v}_{2+n+k} &= - \sum_{j=1}^{n+1} \frac{\tilde{G}_j \eta_k}{(\eta_k + \tilde{\gamma}_j)(1 + \tilde{\gamma}_j)}, \quad k = 1, \dots, m,
\end{aligned} \tag{4.13}$$

and $\tilde{v}_j = 0$ for $3 + n + m \leq j \leq 2m + 2n + 4$.

5. Numerical examples

In this section, some numerical examples are given to illustrate the results obtained in section 4. Following section 4, we consider a variable annuity with guaranteed maturity payment $\max\{F_0, F_T\}$ under the multi-layer expense strategy (1.2) with $B_1 = F_0$ and $B_2 > F_0$, where F_t is determined by (2.1), (2.6) and (2.7). For the sake of simplicity, we consider the case that $m = n = 1$ in (2.2), which means that X in (2.1) is a double exponential jump diffusion process. In the following, for this variable annuity, we will compute its fair fee rates α_1^* and α_2^* via (4.1), (4.9) and (4.12). Besides, with the obtained α_1^* and α_2^* , numerical results on the total collected fees (see (4.3)) and the total time of deducting fees (see Remark 3.3) are also presented.

For numerical computing the above quantities through Laplace inversion, we choose the Euler inversion algorithm, which is first developed in Dubner and Abate (1968) and can be implemented easily. We remark that many papers use this algorithm and its extensions to do numerical Laplace inversion, see, e.g., Petrella (2004). For the convenience of the reader only, we give some important results on this algorithm. For a real function $f(\cdot)$ defined in $(0, \infty)$ and $T \neq 0$,

$$f(T) = \frac{e^{\tilde{A}/2}}{2T} \text{Re}(\hat{f}(\frac{\tilde{A}}{2T})) + \frac{e^{\tilde{A}/2}}{T} \sum_{k=1}^{\infty} (-1)^k \text{Re}(\hat{f}(\frac{\tilde{A} + 2k\pi i}{2T})) - e_d, \quad (5.1)$$

where $\hat{f}(\cdot)$ is the Laplace transform of $f(\cdot)$, \tilde{A} is a positive constant, e_d is the discretization errors and $\text{Re}(x)$ means the real part of x . Moreover, if $|f(T)| \leq B$, then $|e_d| \leq Be^{-\tilde{A}}$; if $|f(T)| \leq BT$, then $|e_d| \leq 3BT e^{-\tilde{A}}$ (see (5.29) in Abate and Whitt (1992)).

Note that, as functions of T , $\max\{\mathbb{E}[e^{-rT}F_T], \mathbb{E}[e^{-rT}(F_0 - F_T)_+]\} \leq F_0$ and $\max\left\{\mathbb{E}\left[\int_0^T \mathbf{1}_{\{U_t < b_1\}} dt\right], \mathbb{E}\left[\int_0^T \mathbf{1}_{\{U_t \geq b_2\}} dt\right]\right\} \leq T$. Therefore, in the following numerical calculations, we set $\tilde{A} = 20$, which is enough to control the discretization errors. Under the martingale measure, the values of the parameters unless stated otherwise are given in Table 1. We remind the reader that

Table 1: Values of the parameters.

parameter	F_0	σ	λ	p_1	q_1	η_1	ϑ_1	r
value	100	0.2	1	0.5	0.5	15	15	0.05

the value of the parameter μ is computed such that $\psi(1) = r$ (the martingale condition), i.e., $\mu = r - \frac{\sigma^2}{2} - \lambda \left(\frac{p_1 \eta_1}{\eta_1 - 1} + \frac{q_1 \vartheta_1}{\vartheta_1 + 1} - 1 \right)$. In addition, all numerical calculations are implemented in MATLAB. For the sake of brevity, the two quantities: $\mathbb{E}\left[\int_0^T \mathbf{1}_{\{U_t < b_1\}} dt\right]$ and $\mathbb{E}\left[\int_0^T \mathbf{1}_{\{U_t \geq b_2\}} dt\right]$ with $b_1 = \ln(\frac{B}{F_0})$ and $b_2 = \ln(\frac{B_2}{F_0})$, are denoted respectively by *Ttime1* and *Ttime2* in the following tables.

Table 2: Fair fee rates α_1^* with respect to B_2 , where $T = 10$ and $\alpha_2^* = \alpha_1^*/2$.

B_2	105	110	120	150	200	300	1000
α_1^*	0.016	0.017	0.018	0.022	0.028	0.038	0.048
Total fees	9.34	9.35	9.47	9.83	10.42	11.85	13.15
Ttime1	4.44	4.43	4.43	4.46	4.56	4.75	4.94
Ttime2	5.03	4.58	3.83	2.31	1.10	0.32	0.002

First, we let $\alpha_2 = \frac{1}{2}\alpha_1$ and study the relationship between the fair fee rate α_1^* and B_2 . We summarize all the related results in Table 2. From the last column of Table 2, we see that the fair fee rate α_1^* for the case of $B_2 = 1000$ is 4.8%, which is too large to be used in practice. In addition, the total time of

deducting fees (Ttime1+Ttime2) is about 4.94. This means that the insurer will have no fees income during 5.06 years (more than half of the variable annuity's maturity) in total, which would be not accepted by the insurer. When B_2 equals 120, the value of α_1^* becomes 1.8% and the total time of deducting fees increases to 8.26. Under this case (i.e., $B_2 = 120$), if the value of F_t exceeds 120, the fee rate imposed to the policyholder is only 0.009 ($\alpha_1^*/2$). Therefore, the strategy that B_2 takes the value of 120 is advisable. Of course, when B_2 decreases, the value of α_1^* also declines. However, from Table 2, we know that the value of α_1^* when $B_2 = 105$ is almost the same as that when $B_2 = 120$. Besides, under these two cases (i.e., $B_2 = 105$ and $B_2 = 120$), the total fees charged by the insurer are nearly the same as well, i.e., the policyholder pays the same cost for the provided guarantee. And because of this, the policyholder will prefer to the case that $B_2 = 120$. So, compared with $B_2 = 120$, the strategy of $B_2 = 105$ has less competitiveness. Therefore, in the following tables (except Table 4), we only consider that $B_2 = 120$.

We are interested in the connection of the fair fee rates α_2^* , α_1^* and the maturity T . The corresponding results are given in Table 3, where once again we let α_2 equal $\alpha_1/2$.

Table 3: Fair fee rates α_1^* with respect to T , where $B_2 = 120$ and $\alpha_2^* = \alpha_1^*/2$.

T	1	3	5	7	10	12	15
α_1^*	0.366	0.098	0.051	0.031	0.018	0.013	0.009
Total fees	21.26	15.82	13.53	11.46	9.47	8.20	7.08
Ttime1	0.68	1.66	2.54	3.32	4.43	5.11	6.11
Ttime2	0.07	0.60	1.37	2.29	3.83	4.94	6.67

Intuitively, when the maturity T becomes large, the total time of deducting fees also grows, which yields a small fee rate. This intuition is confirmed by the second row of Table 3. We remark that the fact that the fair fee rate α_1^* decreases with T encourages the policyholders to hold longer variable annuities in some sense. Besides, unlike the usual financial options (whose prices increase with their maturity), the cost of the guarantees embedded in VAs (i.e., the Total fees in Table 3) decreases with the maturity. One possible reason for this difference is that the cost for a financial option is charged at the beginning of the contract while that for the guaranteed benefit of a variable annuity is paid during the whole life of the policy. Of course, the insurer cannot deduct all costs at the inception of a variable annuity. Otherwise, the insured will pay too much money due to the long maturity. In return, for a financial option (whose maturity is typically lower than one year), one cannot take the fee deducting method used in variable annuities, because the fair fee rate α_1^* amounts to 36.6 percent when $T = 1$ in Table 3. Moreover, numerical results in Table 4 also demonstrate that the fee rates are too large even when $B_2 = 100.1$ and $\alpha_2 = \alpha_1$ (note that the case of $B_2 = 100.1$ and $\alpha_2 = \alpha_1$ means that one charges fees by a fixed rate on matter what the value of F_t is).

In short, short-term contracts and long-term contracts should be treated

Table 4: Fair fee rates α_1^* with respect to B_2 , where $T = 1$.

$\alpha_2^* = 0.5\alpha_1^*$				$\alpha_2^* = \alpha_1^*$			
B_2	100.1	110	120	B_2	100.1	110	120
α_1^*	0.206	0.282	0.366	α_1^*	0.131	0.197	0.291
Total fees	15.09	17.49	21.26	Total fees	12.23	13.74	17.61
Ttime1	0.652	0.646	0.676	Ttime1	0.626	0.605	0.641
Ttime2	0.345	0.159	0.070	Ttime2	0.370	0.165	0.067

separately, and it is likely that a pricing or hedging approach, which is proper to short-term contracts, may be not suitable for long-term contracts.

Table 5: Fair fee rates α_1^* with respect to σ and r , where $B_2 = 120$, $T = 10$ and $\alpha_2^* = \alpha_1^*/2$.

σ	0.1	0.15	0.2	0.25	0.3
α_1^*	0.005	0.011	0.018	0.027	0.036
Total fees	2.35	5.57	9.47	14.37	19.08
r	0.4	0.45	0.5	0.55	0.6
α_1^*	0.026	0.021	0.018	0.015	0.013
Total fees	13.84	11.13	9.47	7.85	6.75

In Table 5, we consider how the fair fee rates depend on the volatility σ and the risk free rate r , respectively. It is obvious that the higher (lower) the volatility σ (the risk free rate r) is, the larger the fee rate α_1^* is. In addition, compared with the risk free rate r , the volatility σ has a larger influence on the fee rate α_1^* . It should be noted that the total fees decrease with the volatility σ . Especially, when $\sigma = 0.1$, the total fees are too small (just 2.35) and only account for 2.35% of the initial premium. This result suggests that the more risk aversion the insured is, the less money he/she pays.

Table 6: Fair fee rates α_1^* with respect to η_1 and ϑ_1 , where $B_2 = 120$, $T = 10$ and $\alpha_2^* = \alpha_1^*/2$.

η_1	6	8	10	15	20	50
α_1^*	0.028	0.023	0.020	0.018	0.017	0.016
Total fees	15.13	12.29	10.62	9.47	8.92	8.36
ϑ_1	6	8	10	15	20	50
α_1^*	0.026	0.022	0.020	0.018	0.017	0.016
Total fees	13.58	11.56	10.52	9.47	8.95	8.41

Finally, in Table 6, we give the results about the sensitivity of the rate α_1 with respect to changes in jump densities η_1 and ϑ_1 . From Table 6, one can see that the fair fee rate α_1^* is decrease with both ϑ_1 and η_1 . When the value of η_1 rises from 6 to 50, the total fees decline from 15.13 to 8.36. Thus the parameter η_1 has a significant effect on the total fees. Besides, one can draw a similar conclusion for the parameter ϑ_1 from Table 6. In short, the jump risk has a large influence on the pricing of variable annuities and thus should be treated

seriously.

6. Conclusion

In this paper, we have investigated the problem of pricing variable annuities with a multi-layer expense strategy. In theory, we have derived formulas for the Laplace transform of the distribution of a jump diffusion process with hyper-exponential jumps and three-valued drift. Applying these formulas, we compute the fair fee rate for a variable annuity with guaranteed minimum maturity benefit under the multi-layer expense strategy via Laplace inversion. Moreover, the total fees and the total time of charging fees are calculated as well. From the numerical results, we find that the fair fee rate is sensitive to the volatility and the risk-free rate. Therefore, a more interesting and challenging extension is considering the case that both the interest rate and the volatility are allowed to be stochastic. Such extension is left for future research.

Appendix A.

The matrix Q_1 in (3.6) is given by

$$Q_1 = \begin{pmatrix} Q_{11} & Q_{12} \end{pmatrix}, \quad (\text{A.1})$$

where

$$Q_{11} = \begin{pmatrix} 1 & \tilde{\beta}_1 & \frac{\vartheta_1}{\vartheta_1 + \beta_1} & \cdots & \frac{\vartheta_n}{\vartheta_n + \beta_1} & \frac{\eta_1}{\eta_1 - \beta_1} & \cdots & \frac{\eta_m}{\eta_m - \beta_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \tilde{\beta}_{m+1} & \frac{\vartheta_1}{\vartheta_1 + \beta_{m+1}} & \cdots & \frac{\vartheta_n}{\vartheta_n + \beta_{m+1}} & \frac{\eta_1}{\eta_1 - \beta_{m+1}} & \cdots & \frac{\eta_m}{\eta_m - \beta_{m+1}} \\ -L^{\beta_1} & -\beta_1 L^{\beta_1} & \frac{-\vartheta_1 L^{\beta_1}}{\vartheta_1 + \beta_1} & \cdots & \frac{-\vartheta_n L^{\beta_1}}{\vartheta_n + \beta_1} & \frac{\eta_1 L^{\beta_1}}{\beta_1 - \eta_1} & \cdots & \frac{\eta_m L^{\beta_1}}{\beta_1 - \eta_m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -L^{\beta_{m+1}} & -\beta_{m+1} L^{\beta_{m+1}} & \frac{-\vartheta_1 L^{\beta_{m+1}}}{\vartheta_1 + \beta_{m+1}} & \cdots & \frac{-\vartheta_n L^{\beta_{m+1}}}{\vartheta_n + \beta_{m+1}} & \frac{\eta_1 L^{\beta_{m+1}}}{\beta_{m+1} - \eta_1} & \cdots & \frac{\eta_m L^{\beta_{m+1}}}{\beta_{m+1} - \eta_m} \\ -1 & \gamma_1 & \frac{-\vartheta_1}{\vartheta_1 - \gamma_1} & \cdots & \frac{-\vartheta_n}{\vartheta_n - \gamma_1} & \frac{-\eta_1}{\eta_1 + \gamma_1} & \cdots & \frac{-\eta_m}{\eta_m + \gamma_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & \gamma_{n+1} & \frac{-\vartheta_1}{\vartheta_1 - \gamma_{n+1}} & \cdots & \frac{-\vartheta_n}{\vartheta_n - \gamma_{n+1}} & \frac{-\eta_1}{\eta_1 + \gamma_{n+1}} & \cdots & \frac{-\eta_m}{\eta_m + \gamma_{n+1}} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (\text{A.2})$$

and

$$Q_{12} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & \beta_1 & \frac{\vartheta_1}{\vartheta_1 + \beta_1} & \cdots & \frac{\vartheta_n}{\vartheta_n + \beta_1} & \frac{\eta_1}{\eta_1 - \beta_1} & \cdots & \frac{\eta_m}{\eta_m - \beta_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \beta_{m+1} & \frac{\vartheta_1}{\vartheta_1 + \beta_{m+1}} & \cdots & \frac{\vartheta_n}{\vartheta_n + \beta_{m+1}} & \frac{\eta_1}{\eta_1 - \beta_{m+1}} & \cdots & \frac{\eta_m}{\eta_m - \beta_{m+1}} \\ L^{\gamma_1} & -\gamma_1 L^{\gamma_1} & \frac{\vartheta_1 L^{\gamma_1}}{\vartheta_1 - \gamma_1} & \cdots & \frac{\vartheta_n L^{\gamma_1}}{\vartheta_n - \gamma_1} & \frac{\eta_1 L^{\gamma_1}}{\eta_1 + \gamma_1} & \cdots & \frac{\eta_m L^{\gamma_1}}{\eta_m + \gamma_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ L^{\gamma_{n+1}} & -\gamma_{n+1} L^{\gamma_{n+1}} & \frac{\vartheta_1 L^{\gamma_{n+1}}}{\vartheta_1 - \gamma_{n+1}} & \cdots & \frac{\vartheta_n L^{\gamma_{n+1}}}{\vartheta_n - \gamma_{n+1}} & \frac{\eta_1 L^{\gamma_{n+1}}}{\eta_1 + \gamma_{n+1}} & \cdots & \frac{\eta_m L^{\gamma_{n+1}}}{\eta_m + \gamma_{n+1}} \\ -1 & \hat{\gamma}_1 & \frac{-\vartheta_1}{\vartheta_1 - \gamma_1} & \cdots & \frac{-\vartheta_n}{\vartheta_n - \gamma_1} & \frac{-\eta_1}{\eta_1 + \gamma_1} & \cdots & \frac{-\eta_m}{\eta_m + \gamma_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & \hat{\gamma}_{n+1} & \frac{-\vartheta_1}{\vartheta_1 - \gamma_{n+1}} & \cdots & \frac{-\vartheta_n}{\vartheta_n - \gamma_{n+1}} & \frac{-\eta_1}{\eta_1 + \gamma_{n+1}} & \cdots & \frac{-\eta_m}{\eta_m + \gamma_{n+1}} \end{pmatrix}, \quad (\text{A.3})$$

with $L = e^{b_1 - b_2}$.

Appendix B. The proof of Theorem 3.1

Before starting the derivation of Theorem 3.1, some notation is introduced first. We set $\tilde{Y} = \{\tilde{Y}_t := X_t - \alpha_1 t; t \geq 0\}$ and $\hat{Y} = \{\hat{Y}_t := X_t - \alpha_2 t; t \geq 0\}$. The law of \tilde{Y} (\hat{Y}) starting from y and the corresponding expectation are denoted by $\tilde{\mathbb{P}}_y$ ($\hat{\mathbb{P}}_y$) and $\tilde{\mathbb{E}}_y$ ($\hat{\mathbb{E}}_y$), respectively. We write briefly $\tilde{\mathbb{P}}$ ($\hat{\mathbb{P}}$) and $\tilde{\mathbb{E}}$ ($\hat{\mathbb{E}}$) when $y = 0$. For $z \in (-\vartheta_1, \eta_1)$, the Lévy exponents of \tilde{Y} and \hat{Y} are given respectively by $\tilde{\psi}(z)$ and $\hat{\psi}(z)$ in (3.3). For any $c, C \in \mathbb{R}$, we define the following stopping times:

$$\begin{aligned} \tau_c^- &:= \inf\{t \geq 0 : X_t \leq c\}, & \tau_C^+ &:= \inf\{t \geq 0 : X_t \geq C\}, \\ \tilde{\tau}_c^- &:= \inf\{t \geq 0 : \tilde{Y}_t \leq c\}, & \tilde{\tau}_C^+ &:= \inf\{t \geq 0 : \tilde{Y}_t \geq C\}, \\ \hat{\tau}_c^- &:= \inf\{t \geq 0 : \hat{Y}_t \leq c\}, & \hat{\tau}_C^+ &:= \inf\{t \geq 0 : \hat{Y}_t \geq C\}, \\ \kappa_c^- &:= \inf\{t \geq 0 : U_t \leq c\}, & \kappa_C^+ &:= \inf\{t \geq 0 : U_t \geq C\}. \end{aligned} \quad (\text{B.1})$$

Besides, we recall the results on the solutions of one-sided and two-sided exit problems of X , \tilde{Y} and \hat{Y} in the following lemma. For their proofs, one can refer to, e.g., Yin et al. (2013) (see Lemmas 2.2, 2.4 and 2.5).

Lemma B.1. (i) Consider any nonnegative measurable function g such that $\int_{-\infty}^0 g(c+y)e^{\vartheta_j y} dy < \infty$ for $j = 1, 2, \dots, n$. For $q > 0$ and $x > c$, we have

$$\hat{\mathbb{E}}_x \left[e^{-q\hat{\tau}_c^-} g(\hat{Y}_{\hat{\tau}_c^-}) \right] = (g(c), g_{\vartheta_1}(c), \dots, g_{\vartheta_n}(c)) \hat{Q}^{-1} \begin{pmatrix} e^{-\hat{\gamma}_1(x-c)} \\ \vdots \\ e^{-\hat{\gamma}_{n+1}(x-c)} \end{pmatrix}, \quad (\text{B.2})$$

where

$$\hat{Q} = \begin{pmatrix} 1 & \frac{\vartheta_1}{\vartheta_1 - \hat{\gamma}_1} & \cdots & \frac{\vartheta_n}{\vartheta_n - \hat{\gamma}_1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{\vartheta_1}{\vartheta_1 - \hat{\gamma}_{n+1}} & \cdots & \frac{\vartheta_n}{\vartheta_n - \hat{\gamma}_{n+1}} \end{pmatrix}, \quad (\text{B.3})$$

and $g_{\vartheta_j}(c) = \int_{-\infty}^0 g(c+y) \vartheta_j e^{\vartheta_j y} dy$ for $j = 1, 2, \dots, n$.

(ii) Consider any nonnegative measurable function g such that $\int_0^\infty g(C+y) e^{-\eta_i y} dy < \infty$ for $i = 1, 2, \dots, m$. For $q > 0$ and $x < C$, we have

$$\tilde{\mathbb{E}}_x \left[e^{-q\tilde{\tau}_C^+} g(\tilde{Y}_{\tilde{\tau}_C^+}) \right] = (g(C), g_{\eta_1}(C), \dots, g_{\eta_m}(C)) \tilde{Q}^{-1} \begin{pmatrix} e^{\tilde{\beta}_1(x-C)} \\ \vdots \\ e^{\tilde{\beta}_{m+1}(x-C)} \end{pmatrix}, \quad (\text{B.4})$$

where

$$\tilde{Q} = \begin{pmatrix} 1 & \frac{\eta_1}{\eta_1 - \beta_1} & \cdots & \frac{\eta_m}{\eta_m - \beta_1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{\eta_1}{\eta_1 - \beta_{m+1}} & \cdots & \frac{\eta_m}{\eta_m - \beta_{m+1}} \end{pmatrix}, \quad (\text{B.5})$$

and $g_{\eta_i}(C) = \int_0^\infty \eta_i e^{-\eta_i y} g(C+y) dy$ for $i = 1, 2, \dots, m$.

(3) Consider any nonnegative measurable function g such that $\int_0^\infty g(C+y) e^{-\eta_i y} dy < \infty$ for $i = 1, 2, \dots, m$, and $\int_{-\infty}^0 g(c+y) e^{\vartheta_j y} dy < \infty$ for $j = 1, 2, \dots, n$. For $q > 0$ and $c < x < C$, we have

$$\mathbb{E}_x \left[e^{-q\tau} g(X_\tau) \right] = (g(C), g_{\eta_1}(C), \dots, g_{\eta_m}(C), g(c), g_{\vartheta_1}(c), \dots, g_{\vartheta_n}(c)) Q_{c,C}^{-1} R_{c,C} \quad (\text{B.6})$$

where $\tau = \min\{\tau_c^-, \tau_C^+\}$, the transpose of the vector $R_{c,C}$ is given by

$$R_{c,C}^T = \left(e^{\beta_1(x-C)}, \dots, e^{\beta_{m+1}(x-C)}, e^{-\gamma_1(x-c)}, \dots, e^{-\gamma_{n+1}(x-c)} \right), \quad (\text{B.7})$$

and

$$Q_{c,C} = \begin{pmatrix} 1 & \frac{\eta_1}{\eta_1 - \beta_1} & \cdots & \frac{\eta_m}{\eta_m - \beta_1} & \bar{x}^{\beta_1} & \frac{\vartheta_1 \bar{x}^{\beta_1}}{\vartheta_1 + \beta_1} & \cdots & \frac{\vartheta_n \bar{x}^{\beta_1}}{\vartheta_n + \beta_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{\eta_1}{\eta_1 - \beta_{m+1}} & \cdots & \frac{\eta_m}{\eta_m - \beta_{m+1}} & \bar{x}^{\beta_{m+1}} & \frac{\vartheta_1 \bar{x}^{\beta_{m+1}}}{\vartheta_1 + \beta_{m+1}} & \cdots & \frac{\vartheta_n \bar{x}^{\beta_{m+1}}}{\vartheta_n + \beta_{m+1}} \\ \bar{x}^{\gamma_1} & \frac{\eta_1 \bar{x}^{\gamma_1}}{\eta_1 + \gamma_1} & \cdots & \frac{\eta_m \bar{x}^{\gamma_1}}{\eta_m + \gamma_1} & 1 & \frac{\vartheta_1}{\vartheta_1 - \gamma_1} & \cdots & \frac{\vartheta_n}{\vartheta_n - \gamma_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{x}^{\gamma_{n+1}} & \frac{\eta_1 \bar{x}^{\gamma_{n+1}}}{\eta_1 + \gamma_{n+1}} & \cdots & \frac{\eta_m \bar{x}^{\gamma_{n+1}}}{\eta_m + \gamma_{n+1}} & 1 & \frac{\vartheta_1}{\vartheta_1 - \gamma_{n+1}} & \cdots & \frac{\vartheta_n}{\vartheta_n - \gamma_{n+1}} \end{pmatrix}, \quad (\text{B.8})$$

with $\bar{x} = e^{c-C}$.

Moreover, the expressions for the distributions of $\hat{S}_{e(q)} := \sup_{0 \leq t \leq e(q)} \hat{Y}_t$ and $\hat{I}_{e(q)} := \inf_{0 \leq t \leq e(q)} \hat{Y}_t$ are also required and are given in the following Lemma B.2. For its derivation, we refer to Lemma 1 in Asmussen et al. (2004).

Lemma B.2. (1) For $s > 0$,

$$\mathbb{E} \left[e^{-s\hat{S}_{e(q)}} \right] = \prod_{i=1}^m \left(\frac{s + \eta_k}{\eta_k} \right) \prod_{k=1}^{m+1} \left(\frac{\hat{\beta}_k}{s + \hat{\beta}_k} \right) = \sum_{k=1}^{m+1} \frac{\hat{C}_k}{s + \hat{\beta}_k}, \quad (\text{B.9})$$

and

$$\hat{\mathbb{P}} \left(\hat{S}_{e(q)} \in dy \right) = \sum_{i=1}^{m+1} \hat{C}_i e^{-\hat{\beta}_i y} dy, \quad y \geq 0, \quad (\text{B.10})$$

where

$$\frac{\hat{C}_i}{\hat{\beta}_i} = \prod_{k=1}^m \left(\frac{\eta_k - \hat{\beta}_i}{\eta_k} \right) \prod_{k=1, k \neq i}^{m+1} \left(\frac{\hat{\beta}_k}{\hat{\beta}_k - \hat{\beta}_i} \right), \quad i = 1, \dots, m+1. \quad (\text{B.11})$$

(2) For $s > 0$ and $y \geq 0$,

$$\mathbb{E} \left[e^{s\hat{I}_{e(q)}} \right] = \prod_{i=1}^m \left(\frac{s + \vartheta_k}{\vartheta_k} \right) \prod_{k=1}^{n+1} \left(\frac{\hat{\gamma}_k}{s + \hat{\gamma}_k} \right) = \sum_{j=1}^{n+1} \frac{\hat{D}_j}{s + \hat{\gamma}_j}, \quad (\text{B.12})$$

and

$$\hat{\mathbb{P}} \left(-\hat{I}_{e(q)} \in dy \right) = \sum_{j=1}^{n+1} \hat{D}_j e^{-\hat{\gamma}_j y} dy, \quad (\text{B.13})$$

where

$$\frac{\hat{D}_j}{\hat{\gamma}_j} = \prod_{k=1}^n \left(\frac{\vartheta_k - \hat{\gamma}_j}{\vartheta_k} \right) \prod_{k=1, k \neq j}^{n+1} \left(\frac{\hat{\gamma}_k}{\hat{\gamma}_k - \hat{\gamma}_j} \right), \quad j = 1, \dots, n+1. \quad (\text{B.14})$$

Remark B.1. For the constants \hat{C}_i in (B.11) and \hat{D}_j in (B.14), we have

$$\sum_{i=1}^{m+1} \frac{\hat{C}_i}{\hat{\beta}_i} = 1 \quad \text{and} \quad \sum_{j=1}^{n+1} \frac{\hat{D}_j}{\hat{\gamma}_j} = 1, \quad (\text{B.15})$$

which can be proved by set $s = 0$ in (B.9) and (B.12). More importantly, as both sides of the second equality in (B.9) are rational function of s , we can extend this identity to the whole complex plane except at $-\hat{\beta}_1, \dots, -\hat{\beta}_{m+1}$. Then, we obtain the following result from the extended identity by letting $s = -\eta_k$:

$$\sum_{i=1}^{m+1} \frac{\hat{C}_i}{\hat{\beta}_i - \eta_k} = 0, \quad \text{for } k = 1, 2, \dots, m. \quad (\text{B.16})$$

Similarly, from (B.12), we have

$$\sum_{j=1}^{n+1} \frac{\hat{D}_j}{\hat{\gamma}_j - \vartheta_k} = 0, \quad \text{for } k = 1, 2, \dots, n. \quad (\text{B.17})$$

Proof. {The proof of Theorem 3.1} This proof is dividend into two steps with the purpose of making it clearly. For the first step, we omit some details as similar arguments have been used in Zhou and Wu (2015).

(i) First, for given $y > b_2 > b_1$, we introduce a function of x as

$$J(x) := \mathbb{P}_x(U_{e(q)} > y). \quad (\text{B.18})$$

For $x < b_1$, we have

$$J(x) = \mathbb{E}_x \left[\mathbf{1}_{\{U_{e(q)} > y\}} \mathbf{1}_{\{e(q) > \kappa_{b_1}^+\}} \right] = \tilde{\mathbb{E}}_x \left[e^{-q\tilde{\tau}_{b_1}^+} J(\tilde{Y}_{\tilde{\tau}_{b_1}^+}) \right]. \quad (\text{B.19})$$

From (B.4), we obtain that

$$J(x) = \sum_{i=1}^{m+1} E_i e^{\tilde{\beta}_i(x-b_1)}, \quad \text{for } x < b_1, \quad (\text{B.20})$$

with

$$(E_1, \dots, E_{m+1}) = (J(b_1), J_{\eta_1}(b_1), \dots, J_{\eta_m}(b_1)) \tilde{Q}^{-1}. \quad (\text{B.21})$$

For $b_1 < x < b_2$, we can derive that

$$\begin{aligned} J(x) &= \mathbb{E}_x \left[e^{-q\kappa_{b_2}^+} \mathbf{1}_{\{\kappa_{b_2}^+ < \kappa_{b_1}^-\}} J(U_{\kappa_{b_2}^+}) \right] + \mathbb{E}_x \left[e^{-q\kappa_{b_1}^-} \mathbf{1}_{\{\kappa_{b_1}^- < \kappa_{b_2}^+\}} J(U_{\kappa_{b_1}^-}) \right] \\ &= \mathbb{E}_x \left[e^{-q\tau_{b_2}^+} \mathbf{1}_{\{\tau_{b_2}^+ < \tau_{b_1}^-\}} J(X_{\tau_{b_2}^+}) \right] + \mathbb{E}_x \left[e^{-q\tau_{b_1}^-} \mathbf{1}_{\{\tau_{b_1}^- < \tau_{b_2}^+\}} J(X_{\tau_{b_1}^-}) \right]. \end{aligned} \quad (\text{B.22})$$

It follows from (B.6) that

$$J(x) = \sum_{i=1}^{m+1} F_i e^{\beta_i(x-b_2)} + \sum_{j=1}^{n+1} G_j e^{\gamma_j(b_1-x)}, \quad \text{for } b_1 < x < b_2, \quad (\text{B.23})$$

with

$$\begin{aligned} (F_1, F_2, \dots, F_{m+1}, G_1, G_2, \dots, G_{n+1}) &= \\ (J(b_2), J_{\eta_1}(b_2), \dots, J_{\eta_m}(b_2), J(b_1), J_{\vartheta_1}(b_1), \dots, J_{\vartheta_n}(b_1)) Q_{b_1, b_2}^{-1}. \end{aligned} \quad (\text{B.24})$$

For $x > b_2$, we can deduce that

$$\begin{aligned} J(x) &= \mathbb{E}_x \left[\mathbf{1}_{\{U_{e(q)} > y\}} \mathbf{1}_{\{e(q) < \kappa_{b_2}^-\}} \right] + \mathbb{E}_x \left[\mathbf{1}_{\{U_{e(q)} > y\}} \mathbf{1}_{\{e(q) > \kappa_{b_2}^-\}} \right] \\ &= \hat{\mathbb{E}}_x \left[\mathbf{1}_{\{\hat{Y}_{e(q)} > y\}} \mathbf{1}_{\{\hat{I}_{e(q)} > b_2\}} \right] + \hat{\mathbb{E}}_x \left[e^{-q\hat{\tau}_{b_2}^-} J(\hat{Y}_{\hat{\tau}_{b_2}^-}) \right] \\ &= \int_{b_2-x}^0 \hat{\mathbb{P}}(\hat{S}_{e(q)} > y-x-z) \hat{\mathbb{P}}(\hat{I}_{e(q)} \in dz) + \hat{\mathbb{E}}_x \left[e^{-q\hat{\tau}_{b_2}^-} J(\hat{Y}_{\hat{\tau}_{b_2}^-}) \right]. \end{aligned} \quad (\text{B.25})$$

Applying (B.2) and Lemma B.2 to (B.25) leads to that

$$J(x) = \sum_{i=1}^{m+1} H_i e^{\hat{\beta}_i(x-y)} + \sum_{j=1}^{n+1} M_j e^{\hat{\gamma}_j(b_2-x)}, \quad \text{for } b_2 < x \leq y, \quad (\text{B.26})$$

and that (note that $\hat{\mathbb{P}}\left(\hat{S}_{e(q)} > t\right) = 1$ for $t \leq 0$)

$$J(x) = 1 + \sum_{j=1}^{n+1} N_j e^{\hat{\gamma}_j(y-x)} + \sum_{j=1}^{n+1} M_j e^{\hat{\gamma}_j(b_2-x)}, \quad \text{for } x \geq y, \quad (\text{B.27})$$

where

$$\begin{aligned} H_i &= \frac{\hat{C}_i}{\hat{\beta}_i} \sum_{j=1}^{n+1} \frac{\hat{D}_j}{\hat{\beta}_i + \hat{\gamma}_j}, \quad i = 1, 2, \dots, m+1, \\ N_j &= \hat{D}_j \sum_{i=1}^{m+1} \frac{\hat{C}_i}{\hat{\beta}_i(\hat{\beta}_i + \hat{\gamma}_j)} - \frac{\hat{D}_j}{\hat{\gamma}_j}, \quad j = 1, 2, \dots, n+1, \end{aligned} \quad (\text{B.28})$$

and

$$\begin{aligned} (M_1, \dots, M_{n+1}) &= - \left(\sum_{i=1}^{m+1} \frac{\hat{D}_1 \hat{C}_i e^{\hat{\beta}_i(b_2-y)}}{\hat{\beta}_i(\hat{\beta}_i + \hat{\gamma}_1)}, \dots, \sum_{i=1}^{m+1} \frac{\hat{D}_{n+1} \hat{C}_i e^{\hat{\beta}_i(b_2-y)}}{\hat{\beta}_i(\hat{\beta}_i + \hat{\gamma}_{n+1})} \right) \\ &\quad + (J(b_2), J_{\vartheta_1}(b_2), \dots, J_{\vartheta_n}(b_2)) \hat{Q}^{-1}. \end{aligned} \quad (\text{B.29})$$

The expression for H_i in (3.5) can be obtained from (B.11), (B.12) and (B.28). Moreover, from (B.28), we have

$$N_j = \hat{D}_j \sum_{i=1}^{m+1} \frac{\hat{C}_i}{\hat{\beta}_i(\hat{\beta}_i + \hat{\gamma}_j)} - \frac{\hat{D}_j}{\hat{\gamma}_j} = \frac{\hat{D}_j}{\hat{\gamma}_j} \sum_{i=1}^{m+1} \hat{C}_i \left(\frac{1}{\hat{\beta}_i} - \frac{1}{(\hat{\beta}_i + \hat{\gamma}_j)} \right) - \frac{\hat{D}_j}{\hat{\gamma}_j}, \quad (\text{B.30})$$

which combined with (B.9) and (B.15), yields the expression of N_j in (3.5). Finally, formula (3.6) will be derived in the second step.

(2) On one hand, from (B.8) and (B.24), we have

$$J_{\vartheta_k}(b_1) = \sum_{i=1}^{m+1} \frac{F_i \vartheta_k}{\vartheta_k + \beta_i} e^{\beta_i(b_1-b_2)} + \sum_{j=1}^{n+1} \frac{G_j \vartheta_k}{\vartheta_k - \gamma_j}, \quad k = 1, 2, \dots, n, \quad (\text{B.31})$$

and

$$J_{\eta_k}(b_2) = \sum_{i=1}^{m+1} \frac{F_i \eta_k}{\eta_k - \beta_i} + \sum_{j=1}^{n+1} \frac{G_j \eta_k}{\eta_k + \gamma_j} e^{\gamma_j(b_1-b_2)}, \quad k = 1, 2, \dots, m. \quad (\text{B.32})$$

On the other hand, applying (B.20) yields

$$J_{\vartheta_k}(b_1) = \int_{-\infty}^0 J(b_1 + y) \vartheta_k e^{\vartheta_k y} dy = \sum_{i=1}^{m+1} \frac{E_i \vartheta_k}{\vartheta_k + \hat{\beta}_i}, \quad k = 1, 2, \dots, n. \quad (\text{B.33})$$

For $1 \leq k \leq m$, using (B.26) and (B.27), we obtain

$$\begin{aligned} J_{\eta_k}(b_2) &= \int_0^\infty J(b_2 + z) \eta_k e^{-\eta_k z} dz = e^{\eta_k(b_2-y)} + \sum_{j=1}^{n+1} \frac{N_j \eta_k}{\eta_k + \hat{\gamma}_j} e^{\eta_k(b_2-y)} \\ &\quad + \sum_{i=1}^{m+1} \frac{H_i \eta_k}{\hat{\beta}_i - \eta_k} \left(e^{\eta_k(b_2-y)} - e^{\hat{\beta}_i(b_2-y)} \right) + \sum_{j=1}^{n+1} \frac{M_j \eta_k}{\eta_k + \hat{\gamma}_j}. \end{aligned} \quad (\text{B.34})$$

Therefore, from (B.31) \sim (B.34), we immediately obtain that

$$\sum_{i=1}^{m+1} \frac{E_i \vartheta_k}{\vartheta_k + \tilde{\beta}_i} = \sum_{i=1}^{m+1} \frac{F_i \vartheta_k}{\vartheta_k + \beta_i} e^{\beta_i(b_1 - b_2)} + \sum_{j=1}^{n+1} \frac{G_j \vartheta_k}{\vartheta_k - \gamma_j}, \quad k = 1, 2, \dots, n, \quad (\text{B.35})$$

and that, for $1 \leq k \leq m$,

$$\sum_{i=1}^{m+1} \frac{F_i \eta_k}{\eta_k - \beta_i} + \sum_{j=1}^{n+1} \frac{G_j \eta_k}{\eta_k + \gamma_j} e^{\gamma_j(b_1 - b_2)} = \sum_{j=1}^{n+1} \frac{M_j \eta_k}{\eta_k + \hat{\gamma}_j} - \sum_{i=1}^{m+1} \frac{H_i \eta_k}{\hat{\beta}_i - \eta_k} e^{\hat{\beta}_i(b_2 - y)}. \quad (\text{B.36})$$

To derive (B.36), we have used the following identity:

$$\sum_{i=1}^{m+1} \frac{H_i \eta_k}{\hat{\beta}_i - \eta_k} + 1 + \sum_{j=1}^{n+1} \frac{N_j \eta_k}{\eta_k + \hat{\gamma}_j} = 0, \quad (\text{B.37})$$

which can be proved as following:

$$\begin{aligned} & \sum_{i=1}^{m+1} \frac{H_i \eta_k}{\hat{\beta}_i - \eta_k} + \sum_{j=1}^{n+1} \frac{N_j \eta_k}{\eta_k + \hat{\gamma}_j} \\ &= \sum_{i=1}^{m+1} \frac{\hat{C}_i}{\beta_i} \sum_{j=1}^{n+1} \frac{\hat{D}_j \eta_k}{\eta_k + \hat{\gamma}_j} \left(\frac{1}{\hat{\beta}_i - \eta_k} - \frac{1}{\hat{\beta}_i + \hat{\gamma}_j} \right) + \sum_{j=1}^{n+1} \frac{N_j \eta_k}{\eta_k + \hat{\gamma}_j} \\ &= \left(\sum_{i=1}^{m+1} \frac{\hat{C}_i}{\hat{\beta}_i - \eta_k} - \sum_{i=1}^{m+1} \frac{\hat{C}_i}{\hat{\beta}_i} \right) \sum_{j=1}^{n+1} \frac{\hat{D}_j \eta_k}{\eta_k + \hat{\gamma}_j} - \sum_{j=1}^{n+1} \frac{\hat{D}_j \eta_k}{\eta_k + \hat{\gamma}_j} = -1, \end{aligned} \quad (\text{B.38})$$

where the second equality follows from (B.28) and the third one is due to (B.15) and (B.16).

Next, it follows from (B.3), (B.5), (B.21) and (B.29) that

$$J_{\eta_k}(b_1) = \sum_{i=1}^{m+1} \frac{E_i \eta_k}{\eta_k - \tilde{\beta}_i}, \quad k = 1, 2, \dots, m, \quad (\text{B.39})$$

and that

$$\begin{aligned} J_{\vartheta_k}(b_2) &= \sum_{j=1}^{n+1} \left(M_j + \sum_{i=1}^{m+1} \frac{\hat{C}_i \hat{D}_j}{\hat{\beta}_i(\hat{\beta}_i + \hat{\gamma}_j)} e^{\hat{\beta}_i(b_2 - y)} \right) \frac{\vartheta_k}{\vartheta_k - \hat{\gamma}_j} \\ &= \sum_{j=1}^{n+1} \left(M_j + \sum_{i=1}^{m+1} \frac{\hat{C}_i \hat{D}_j \vartheta_k}{\hat{\beta}_i(\vartheta_k + \hat{\beta}_i)} \left(\frac{1}{\hat{\beta}_i + \hat{\gamma}_j} + \frac{1}{\vartheta_k - \hat{\gamma}_j} \right) e^{\hat{\beta}_i(b_2 - y)} \right) \\ &= \sum_{j=1}^{n+1} \frac{M_j \vartheta_k}{\vartheta_k - \hat{\gamma}_j} + \sum_{i=1}^{m+1} \frac{H_i \vartheta_k}{\vartheta_k + \hat{\beta}_i} e^{\hat{\beta}_i(b_2 - y)}, \quad k = 1, 2, \dots, n. \end{aligned} \quad (\text{B.40})$$

where the second equality can be verified by using (B.17) and (B.28).

Using (B.20), (B.23), (B.26) and (B.27), one can derive that

$$\begin{aligned} J_{\vartheta_k}(b_2) &= \int_{-\infty}^0 J(b_2 + y) \vartheta_k e^{\vartheta_k y} dy = \sum_{j=1}^{n+1} \frac{G_j \vartheta_k}{\vartheta_k - \gamma_j} \left(e^{\gamma_j(b_1-b_2)} - e^{\vartheta_k(b_1-b_2)} \right) \\ &+ \sum_{i=1}^{m+1} \frac{E_i \vartheta_k}{\vartheta_k + \hat{\beta}_i} e^{\vartheta_k(b_1-b_2)} + \sum_{i=1}^{m+1} \frac{F_i \vartheta_k}{\vartheta_k + \beta_i} \left(1 - e^{(\vartheta_k + \beta_i)(b_1-b_2)} \right), \quad 1 \leq k \leq n, \end{aligned} \quad (\text{B.41})$$

and that, for $k = 1, \dots, m$,

$$\begin{aligned} J_{\eta_k}(b_1) &= \int_0^\infty J(b_1 + z) \eta_k e^{-\eta_k z} dz = \sum_{j=1}^{n+1} \frac{M_j \eta_k}{\eta_k + \hat{\gamma}_j} e^{\eta_k(b_1-b_2)} + e^{\eta_k(b_1-y)} \\ &+ \sum_{i=1}^{m+1} \frac{F_i \eta_k}{\beta_i - \eta_k} \left(e^{\eta_k(b_1-b_2)} - e^{\beta_i(b_1-b_2)} \right) + \sum_{j=1}^{n+1} \frac{G_j \eta_k}{\eta_k + \gamma_j} \left(1 - e^{(\eta_k + \gamma_j)(b_1-b_2)} \right) \\ &+ \sum_{i=1}^{m+1} \frac{H_i \eta_k}{\hat{\beta}_i - \eta_k} \left(e^{\eta_k(b_1-y)} - e^{\eta_k(b_1-b_2)} e^{\hat{\beta}_i(b_2-y)} \right) + \sum_{j=1}^{n+1} \frac{N_j \eta_k}{\eta_k + \hat{\gamma}_j} e^{\eta_k(b_1-y)}. \end{aligned} \quad (\text{B.42})$$

Thus, for $1 \leq k \leq n$, applying formulas (B.35), (B.40) and (B.41) leads to

$$\sum_{j=1}^{n+1} \frac{M_j \vartheta_k}{\vartheta_k - \hat{\gamma}_j} + \sum_{i=1}^{m+1} \frac{H_i \vartheta_k}{\vartheta_k + \hat{\beta}_i} e^{\hat{\beta}_i(b_2-y)} = \sum_{i=1}^{m+1} \frac{F_i \vartheta_k}{\vartheta_k + \beta_i} + \sum_{j=1}^{n+1} \frac{G_j \vartheta_k}{\vartheta_k - \gamma_j} e^{\gamma_j(b_1-b_2)}, \quad (\text{B.43})$$

and for $k = 1, 2, \dots, m$, applying (B.36), (B.37), (B.39) and (B.42) leads to

$$\sum_{i=1}^{m+1} \frac{E_i \eta_k}{\eta_k - \hat{\beta}_i} = \sum_{i=1}^{m+1} \frac{F_i \eta_k}{\eta_k - \beta_i} e^{\beta_i(b_1-b_2)} + \sum_{j=1}^{n+1} \frac{G_j \eta_k}{\eta_k + \gamma_j}. \quad (\text{B.44})$$

Finally, from part (1) of Lemma 2.1, we know that $J(x)$ is continuously differentiable on \mathbb{R} . This means that $J(b_1-) = J(b_1+)$, $J(b_2-) = J(b_2+)$, $J'(b_1-) = J'(b_1+)$ and $J'(b_2-) = J'(b_2+)$, which combined with (B.20), (B.23) and (B.26), yields

$$\begin{aligned} \sum_{i=1}^{m+1} E_i &= \sum_{i=1}^{m+1} F_i e^{\beta_i(b_1-b_2)} + \sum_{j=1}^{n+1} G_j, \\ \sum_{i=1}^{m+1} E_i \hat{\beta}_i &= \sum_{i=1}^{m+1} F_i \beta_i e^{\beta_i(b_1-b_2)} - \sum_{j=1}^{n+1} G_j \gamma_j, \\ \sum_{i=1}^{m+1} F_i + \sum_{j=1}^{n+1} G_j e^{\gamma_j(b_1-b_2)} &= \sum_{i=1}^{m+1} H_i e^{\hat{\beta}_i(b_2-y)} + \sum_{j=1}^{n+1} M_j, \\ \sum_{i=1}^{m+1} F_i \beta_i - \sum_{j=1}^{n+1} G_j \gamma_j e^{\gamma_j(b_1-b_2)} &= \sum_{i=1}^{m+1} H_i \hat{\beta}_i e^{\hat{\beta}_i(b_2-y)} - \sum_{j=1}^{n+1} M_j \hat{\gamma}_j. \end{aligned} \quad (\text{B.45})$$

Therefore, from (B.35), (B.36), (B.43), (B.44) and (B.45), we deduce (3.6). This completes the proof. \square

Appendix C.

Proof. {The proof of Corollary 3.1} It follows from (3.4) and (3.14) that

$$\begin{aligned} \mathbb{P}_x(U_{e(q)} > b_2) - \mathbb{P}_x(U_{e(q)} \geq b_2) &= \lim_{y \downarrow b_2} \mathbb{P}_x(U_{e(q)} > y) - \lim_{y \uparrow b_2} \mathbb{P}_x(U_{e(q)} > y) \\ &= \begin{cases} \sum_{i=1}^{m+1} (E_i^0 - \hat{E}_i^0) e^{\tilde{\beta}_i(x-b_1)}, & x \leq b_1, \\ \sum_{i=1}^{m+1} (F_i^0 - \hat{H}_i^0 - \hat{U}_i) e^{\beta_i(x-b_2)} + \sum_{j=1}^{n+1} (G_j^0 - \hat{G}_j^0) e^{\gamma_j(b_1-x)}, & b_1 \leq x \leq b_2, \\ \sum_{j=1}^{n+1} (N_j + M_j^0 - \hat{N}_j^0) e^{\hat{\gamma}_j(b_2-x)}, & x \geq b_2, \end{cases} \end{aligned} \quad (\text{C.1})$$

with

$$\begin{aligned} (\hat{E}_1^0, \dots, \hat{E}_{m+1}^0, \hat{H}_1^0, \dots, \hat{H}_{m+1}^0, \hat{G}_1^0, \dots, \hat{G}_{n+1}^0, \hat{N}_1^0, \dots, \hat{N}_{n+1}^0) Q_1 &= \hat{h}^0, \\ (E_1^0, \dots, E_{m+1}^0, F_1^0, \dots, F_{m+1}^0, G_1^0, \dots, G_{n+1}^0, M_1^0, \dots, M_{n+1}^0) Q_1 &= h^0, \end{aligned} \quad (\text{C.2})$$

where h^0 and \hat{h}^0 are given by h in (3.7) and \hat{h} in (3.17) with $y = b_2$, respectively.

Similar to derive (B.37), we can obtain the following equalities by using (B.15), (B.17) and (B.28):

$$\begin{aligned} \sum_{i=1}^{m+1} H_i &= 1 + \sum_{j=1}^{n+1} N_j, \\ \sum_{i=1}^{m+1} H_i \hat{\beta}_i + \sum_{j=1}^{n+1} N_j \hat{\gamma}_j &= 0, \\ \sum_{i=1}^{m+1} \frac{H_i \vartheta_k}{\vartheta_k + \hat{\beta}_i} - \sum_{j=1}^{n+1} \frac{N_j \vartheta_k}{\vartheta_k - \hat{\gamma}_j} - 1 &= 0, \quad \text{for } k = 1, 2, \dots, n. \end{aligned} \quad (\text{C.3})$$

From (3.20), (B.37), (C.2), (C.3) and (A.1), we can verify easily that

$$E_i^0 = \hat{E}_i^0, \quad F_i^0 - \hat{H}_i^0 - \hat{U}_i = 0, \quad G_j^0 - \hat{G}_j^0 = 0, \quad N_j + M_j^0 - \hat{N}_j^0 = 0, \quad (\text{C.4})$$

i.e.,

$$(\underbrace{0, \dots, 0}_{m+1}, \hat{U}_1, \dots, \hat{U}_{m+1}, \underbrace{0, \dots, 0}_{n+1}, -N_1, \dots, -N_{n+1}) Q_1 = h^0 - \hat{h}^0. \quad (\text{C.5})$$

Combining (C.1) with (C.4) leads to

$$\mathbb{P}_x(U_{e(q)} = b_2) = 0. \quad (\text{C.6})$$

The proof of $\mathbb{P}_x(U_{e(q)} = b_1) = 0$ is similar, thus we omit the details. \square

Proof. {The proof of Corollary 3.2} For fixed b_1 , letting $b_2 \downarrow b_1$ in (3.4), we immediately deduce (3.23) with $E_i^1 := \lim_{b_2 \downarrow b_1} E_i$ and $M_j^1 := \lim_{b_2 \downarrow b_1} M_j$.

From (B.35) and (B.43) with $b_2 \downarrow b_1$, we can obtain

$$\sum_{j=1}^{n+1} \frac{M_j^1}{\vartheta_k - \hat{\gamma}_j} + \sum_{i=1}^{m+1} \frac{H_i}{\vartheta_k + \hat{\beta}_i} e^{\hat{\beta}_i(b_1-y)} = \sum_{i=1}^{m+1} \frac{E_i^1}{\vartheta_k + \hat{\beta}_i}, \quad 1 \leq k \leq n. \quad (\text{C.7})$$

Besides, it following from (B.36) and (B.44) with $b_2 = b_1$ that

$$\sum_{i=1}^{m+1} \frac{E_i^1}{\eta_k - \hat{\beta}_i} = \sum_{j=1}^{n+1} \frac{M_j^1}{\eta_k + \hat{\gamma}_j} - \sum_{i=1}^{m+1} \frac{H_i}{\hat{\beta}_i - \eta_k} e^{\hat{\beta}_i(b_1-y)}, \quad 1 \leq k \leq m. \quad (\text{C.8})$$

Applying (B.45) with $b_2 = b_1$ produces

$$\begin{aligned} \sum_{i=1}^{m+1} E_i^1 &= \sum_{i=1}^{m+1} H_i e^{\hat{\beta}_i(b_1-y)} + \sum_{j=1}^{n+1} M_j^1, \\ \sum_{i=1}^{m+1} E_i^1 \hat{\beta}_i &= \sum_{i=1}^{m+1} H_i \hat{\beta}_i e^{\hat{\beta}_i(b_1-y)} - \sum_{j=1}^{n+1} M_j^1 \hat{\gamma}_j. \end{aligned} \quad (\text{C.9})$$

Next, we want to solve (C.7), (C.8) and (C.9). First, define a function of x as

$$f(x) = \sum_{i=1}^{m+1} \frac{E_i^1}{x - \hat{\beta}_i} - \sum_{j=1}^{n+1} \frac{M_j^1}{x + \hat{\gamma}_j} - \sum_{i=1}^{m+1} \frac{H_i}{x - \hat{\beta}_i} e^{\hat{\beta}_i(b_1-y)}. \quad (\text{C.10})$$

From (C.7) and (C.8), we will obtain that $f(-\vartheta_k) = 0$ for $1 \leq k \leq n$ and $f(\eta_k) = 0$ for $1 \leq k \leq m$. Combining these results with (C.9), we obtain that

$$f(x) = \frac{\prod_{i=1}^m (x - \eta_i) \prod_{j=1}^n (x + \vartheta_j)}{\prod_{i=1}^{m+1} (x - \hat{\beta}_i) \prod_{j=1}^{n+1} (x + \hat{\gamma}_j)} \frac{l_m x^m + l_{m-1} x^{m-1} + \cdots + l_0}{\prod_{i=1}^{m+1} (x - \hat{\beta}_i)}, \quad (\text{C.11})$$

for some proper constants l_m, l_{m-1}, \dots, l_0 . Furthermore, for $1 \leq i \leq m+1$, it follows from the definition (C.10) that

$$\lim_{x \rightarrow \hat{\beta}_i} f(x)(x - \hat{\beta}_i) = -H_i e^{\hat{\beta}_i(b_1-y)}. \quad (\text{C.12})$$

From (C.11) and (C.12), we conclude that $f(x)$ has another form as following:

$$f(x) = \frac{\prod_{i=1}^m (x - \eta_i) \prod_{j=1}^n (x + \vartheta_j)}{\prod_{i=1}^{m+1} (x - \hat{\beta}_i) \prod_{j=1}^{n+1} (x + \hat{\gamma}_j)} \sum_{i=1}^{m+1} \frac{\prod_{k=1}^{m+1} (\hat{\beta}_i - \hat{\beta}_k) \prod_{k=1}^{n+1} (\hat{\beta}_i + \hat{\gamma}_k)}{\prod_{k=1}^m (\hat{\beta}_i - \eta_k) \prod_{k=1}^n (\hat{\beta}_i + \vartheta_k)} \frac{-H_i}{x - \hat{\beta}_i} e^{\hat{\beta}_i(b_1-y)}. \quad (\text{C.13})$$

Therefore, from (C.10), (C.13) and the expression of H_i in (3.5), we can derive (3.24) and (3.25). The derivation of (3.26) from Theorem 3.2 is very similar, thus we omit the details. For (3.29), one can obtain it by using a similar

idea to that in the proof of Corollary 3.1. However, as this process involves some computations, we give the details for the convenience of the reader.

From (3.23) and (3.26), we can show that

$$\begin{aligned}
& \mathbb{P}_x(U_{e(q)} > b_1) + \mathbb{P}_x(U_{e(q)} < b_1) \\
&= \lim_{y \downarrow b_1} \mathbb{P}_x(U_{e(q)} > y) + \lim_{y \uparrow b_1} \mathbb{P}_x(U_{e(q)} < y) \\
&= \begin{cases} 1 + \sum_{i=1}^{m+1} (\tilde{E}_i + \tilde{F}_i^{1,0} + E_i^{1,0}) e^{\tilde{\beta}_i(x-b_1)}, & x \leq b_1, \\ 1 + \sum_{j=1}^{n+1} (\tilde{N}_j^{1,0} + N_j + M_j^{1,0}) e^{\tilde{\gamma}_j(b_1-x)}, & x \geq b_1, \end{cases} \quad (C.14)
\end{aligned}$$

where $M_j^{1,0}$, $E_i^{1,0}$, $\tilde{N}_j^{1,0}$ and $\tilde{F}_i^{1,0}$ are given respectively by M_j^1 , E_i^1 , \tilde{N}_j^1 and \tilde{F}_i^1 with $y = b_1$. From (3.5), (3.10), (3.24), (3.25), (3.27) and (3.28), we can obtain the following result by using Lemma C.1:

$$\tilde{E}_i + \tilde{F}_i^{1,0} + E_i^{1,0} = 0, \quad \tilde{N}_j^{1,0} + N_j + M_j^{1,0} = 0. \quad (C.15)$$

Therefore, formulas (C.14) and (C.15) lead to (3.29). \square

Lemma C.1. *For distinct constants $\tilde{l}_1, \dots, \tilde{l}_{n_1}$ and arbitrary constants $\hat{l}_1, \dots, \hat{l}_{m_1}$ with $m_1 < n_1 - 1$, we have*

$$\sum_{i=1}^{n_1} \frac{\prod_{k=1}^{m_1} (\tilde{l}_i - \hat{l}_k)}{\prod_{k=1, k \neq i}^{n_1} (\tilde{l}_i - \tilde{l}_k)} = 0. \quad (C.16)$$

Proof. The proof is easy by noting that the left-hand side of (C.16) is the coefficient of x^{n_1-1} in the numerator of the following rational function:

$$\frac{\prod_{i=1}^{m_1} (x - \hat{l}_i)}{\prod_{i=1}^{n_1} (x - \tilde{l}_i)} = \sum_{i=1}^{n_1} \frac{\prod_{k=1}^{m_1} (\tilde{l}_i - \hat{l}_k)}{\prod_{k=1, k \neq i}^{n_1} (\tilde{l}_i - \tilde{l}_k)} \frac{1}{x - \tilde{l}_i}. \quad (C.17)$$

\square

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